# PLANE GEOMETRY

REVISED EDITION

SCHULTZE-SEVENOAK-SCHUYLER

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# PLANE GEOMETRY

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# PLANE GEOMETRY

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# PREFACE TO THE FIRST EDITION

It is generally conceded that the final aim of mathematical teaching should be not only the acquisition of practical knowledge but that training of the student's mind which gives a distinct gain in mental power. In recognition of this principle nearly all college entrance examinations in geometry require some original work, and most textbooks devote considerable space to exercises. Comparatively little, however, has been done to introduce the student *systematically* to original geometrical work. No teacher of physics or chemistry would ask a student to discover a law without so guiding his work as to enable him to reach the desired result; many textbooks and teachers expect the pupil to invent geometrical proofs and to solve problems, entirely new to him, without offering any assistance further than a knowledge of the well-established theorems of all textbooks. Some writers give a description of the analysis of propositions, which is entirely logical and of great advantage to a person of some mathematical knowledge, but which is usually too abstract to be of any practical value to the beginner. In this book the attempt is made to introduce the student systematically to the solution of geometrical exercises. In the beginning the exercises given in a certain group are of similar kind and related to the preceding proposition; later some general principles are developed which are of fundamental importance for original work, as, for example, the method of proving the equality of lines by means of equal triangles; the method of proving the proportionality of lines by means of similar triangles, etc.; and

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finally the analyses of theorems and problems are introduced, but in a more concrete form than usual.

The propositions are arranged with the view to obtaining a perfect logical and pedagogical order. An unusually large number of exercises is given, selected with care for the purpose of securing increased mental power.

The general plan and the preparation of the greater part of the book are the work of Dr. Schultze, while that of Dr. Sevenoak has been chiefly editorial.

## PREFACE TO THE FIRST REVISED EDITION

The main purpose of the revision of this book has been to emphasize still further and to elaborate in greater detail the principal aim of the original edition, viz., *to introduce the student systematically to original geometric work.* To make the teaching of geometry both disciplinary and informational; to give to the student mental training instead of teaching him mere facts; to develop his power instead of making him memorize, — these are the fundamental aims of this book.

The means employed for this purpose are similar to those used in the first edition. Still greater emphasis, however, has been placed upon the general methods which may be used for the solution of original exercises. The grading and the selection of exercises have been carefully revised. All originals that appeared unfit or too difficult have been eliminated or replaced by simpler and better ones. Topics of fundamental importance, *e.g.* the methods of demonstrating the equality of lines, are represented in greater detail and illustrated by a greater number of exercises than in the first edition.

In addition to these fundamental tendencies, a number of minor improvements have been introduced, among which may be mentioned:

*Improved presentation of the regular propositions.* Many proofs have been simplified, a more pedagogic sequence of the propositions of Book I has been adopted, Books VI and VII have been considerably simplified, and a number of difficult theorems of minor importance have been omitted or placed in the appendix.

*Simplification of the so-called "incommensurable case."* As this is a claim that is made by most textbooks, it may be received with some degree of skepticism, but a repeated trial of this new method will reveal its simplicity. For the more conservative teacher, however, who dislikes fundamental changes, the time-honored method is given in the appendix.

*The introduction of many applied problems.* These problems have been selected and arranged so as to increase the interest of the student, without sacrificing in the least the disciplinary value of the subject. Many such problems are given in the appendix.

*The arrangement of the propositions and the terminology are in accord with the best modern usage.* Thus, statements and reasons have been separated and placed in parallel vertical columns; the term "congruent" and the corresponding symbol are introduced and applied consistently, etc.

*Many of the diagrams have been improved.* The construction lines are drawn completely for most problems, graphical modes are employed for pointing out important facts, and many diagrams have been otherwise improved.

Thanks are due to Dr. J. Kahn and Mr. W. S. Schlauch for assistance in reading the proof and for helpful suggestions.

A. S.

August 1, 1913.

# PREFACE TO THE PRESENT EDITION

In the twelve years since the first revision of Schultze and Sevenoak's Geometry was made, the progress of teaching and the codifying of mathematical requirements have necessitated some changes in form and content. The present revision follows closely the first revision in organization, but has been completely rewritten with additional material. The noteworthy points are:

1. The text has been made to conform to the "Report of the National Committee on Mathematical Requirements" and to the requirements of the New York State Regents and of the College Entrance Examination Board.

2. Most of the excellent exercises of the previous edition have been kept intact and other exercises, from recent examination questions, have been added.

3. The propositions, especially in Books I and II, have been rather fully developed.

4. Additions have been made in (1) propositions, (2) simple trigonometric functions, (3) tables, (4) short review section on arithmetic and algebra, (5) a sketch of the history of geometry.

5. At the end of the various books and in the appendix, the more difficult propositions have been retained so as to meet the most exacting requirements of any college or technical school.

The authors and publishers are conscious of a great debt of gratitude to many teachers who have aided in this revision by their criticisms and suggestions. While a full list of such collaborators is impossible, special thanks are due the following: Mr. Stephen Emery, Erasmus Hall High School, Dr. Loring B. Mullen, Girls High School, Miss Josephine D. Wilkin, Jamaica High School, Mr. Philip R. Dean, Evander Childs High School, and Mrs. Jean F. Brown, Hunter College High School, of New York City; Mr. Edmund D. Searls, High School, of New Bedford, Massachusetts; Miss A. Laura Batt, High School, of Somerville, Massachusetts: Mr. Allen H. Knapp, Central High School, and Mr. Harry B. Marsh, Technical High School, of Springfield, Massachusetts; and Miss Anna H. Andrews, Public High School, of Hartford, Connecticut.

E. S.

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# SUGGESTIONS TO STUDENTS

The student should set to work guided by the following suggestions:

1. Have materials — pencil, paper, ruler, compasses — ready for use.

2. Have a definite time and place to do your work.

3. Do your work *yourself*. Do it *regularly* and *conscientiously*.

4. Frequently review important matters already gone over to fix them firmly in your mind.

5. In a notebook classify your results, stating conditions that make angles equal, lines parallel, triangles equal, etc. Classify formulas.

6. Do not merely skim through the text. Give it thought. Read carefully, and thoroughly digest what you do.

# SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| $+$ | plus, or added to. |
| $-$ | minus, or diminished by. |
| $=$ | equals, or is equivalent to. |
| $\cong$ | congruent. |
| $\neq$ | is not equal to. |
| $>$ | is greater than. |
| $<$ | is less than. |
| $\therefore$ | therefore, or hence. |
| $\perp$ | perpendicular, or is perpendicular to. |
| $\perp$s | perpendiculars. |
| $\parallel$ | parallel, or is parallel to. |
| $\parallel$s | parallels. |
| $\sim$ | is similar to, or similar. |
| $\angle$ | angle. |
| $\angle$s | angles. |
| $\triangle$ | triangle. |
| $\triangle$s | triangles. |
| ▱ | parallelogram. |
| ▱s | parallelograms. |
| $\odot$ | circle. |
| ⓢ | circles. |
| $\frown$ | arc; as $\overset{\frown}{AB}$, arc $AB$. |
| adj. | adjacent. |

| | |
|---|---|
| alt. | alternate. |
| ax. | axiom. |
| circum. | circumference. |
| comp. | complement. |
| con. | construction. |
| cor. | corollary. |
| corr. | corresponding. |
| def. | definition. |
| ex. | exercise. |
| ext. | exterior. |
| hom. | homologous. |
| hy. | hypotenuse. |
| hyp. | hypothesis. |
| iden. | identity. |
| int. | interior. |
| isos. | isosceles. |
| rt. | right. |
| st. | straight. |
| sub. | substitution. |
| sup. | supplementary, or supplement. |

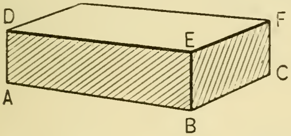
# PLANE GEOMETRY

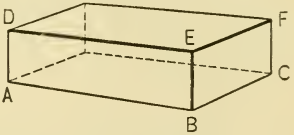
# PLANE GEOMETRY

## INTRODUCTION

### DEFINITIONS

**1.** A physical body, such as a block of wood or iron, occupies a definite portion of space. The portion of space occupied by a physical body is called a *geometric solid* or a *solid.*





**2.** Def. **Solids** are limited portions of space. They have three dimensions: *length*, *breadth*, and *thickness.*

**3.** Def. **Surfaces** are the boundaries of solids; as *ABED* or *BEFC*. They have two dimensions: *length* and *breadth.*

The boundary between a window pane and the air is a surface. Obviously such a boundary has no thickness.

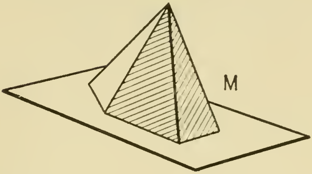
**4.** Def. **Lines** are the boundaries of surfaces; as *AB*, *AD*. (Figure of § 1.) Lines have but one dimension: *length.*

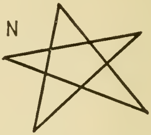
Thus, the annexed black line *AB* is not a geometric line, for it has breadth. A true geometric line, however, is represented by the boundary between the black and the white.

**5.** Def. **Points** are the boundaries, or the extremities, of lines. They are without dimensions, having position only.

Surfaces may be conceived as existing independently of the solids whose boundaries they form. In like manner, lines and points may exist independently in space.

**6.** Def. A **geometric figure** is a point, line, surface, or solid, or any combination of any or all of these; as *M* or *N*.

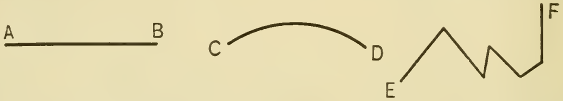




A *rectilinear figure* is a figure composed of straight lines only.

**7.** Def. **Geometry** is the science that treats of the properties of geometric figures.

**8.** The simplest line is a **straight line.** It is *represented* approximately by a string stretched taut between two points; as *AB*. The word *line* is frequently used to denote a straight line.



The notion of a straight line is such a simple and fundamental one that it is practically impossible to give a good definition of it.

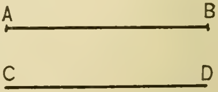
**9.** Def. A **curved line** is a line no portion of which is straight; as *CD* in § 8.

**10.** Def. A **broken line** is a line composed of different successive straight lines; as *EF* in § 8.

No *two successive* parts of a broken line lie in the same straight line.

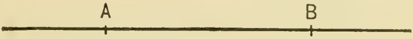
**11.** The expression, straight line, is used to denote both an unlimited straight line and a part of such line.

A *line of definite length,* also called a **segment,** or **line-segment,** is represented by a line whose ends are marked; as *AB*. The length of this line is also called the **distance** between *A* and *B*.



A line whose ends are not marked represents a line of indefinite length; as *CD*.

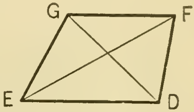
**12.** The **direction** of the line $AB$ means the direction from $A$ toward $B$; of $BA$, the direction from $B$ toward $A$.



**13.** To **produce** the line $AB$ means to prolong it through $B$; to **produce** $BA$ means to prolong it through $A$.

**14.** Def. A **plane surface**, or a **plane**, is a surface such that a straight line joining any two of its points lies entirely in the surface.

**15.** Def. A **plane figure** is a geometric figure, all of whose points lie in the same plane; as $EDFG$.



**16.** Def. **Plane Geometry** treats of plane figures only.

**17.** Def. **Solid Geometry** treats of figures which are not plane. **Spherical Geometry** is a surface geometry.

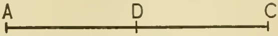
**18.** When one figure can be placed upon another so that every point of each figure lies upon some point of the other, the figures are said to **coincide**.

✓ **19.** Def. **Congruent figures** are those that can be made to coincide.

For reasons that will appear later congruent lines are frequently called *equal lines*. Similarly, congruent angles are usually called *equal angles*. (See note, p. 205.)

**20.** **Proof by superposition** is the method of proving the congruence of two figures by making them coincide.

✓ **21.** To **bisect** a line means to divide it into two equal parts.



Thus, $AC$ is bisected if $AD = DC$. We assume that every line-segment ($AC$) has one and only one bisector. This bisector is a point; as $D$.

Ex. **1.** What is the path of a moving point?

Ex. **2.** What geometric figure is, in general, generated by a moving line? By a moving surface?

Ex. **3.** Can a straight line move so that its path is not a surface?

Ex. **4.** How does a stone cutter use the straight edge to determine whether a surface is plane?

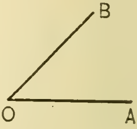
Ex. **5.** What kind of surface is represented by each wall of a room?

Ex. **6.** What kind of surface is represented by the outer surface of a gas pipe?

## ANGLES

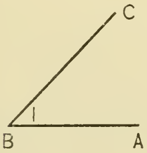
**22.** If a straight line *OA* revolves about one of its points *O* until the line reaches the position *OB*, then the amount of this rotation is called *the angle AOB*. Obviously the amount of rotation, and hence the angle, does not depend upon the length of the line which rotates.

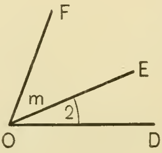
The lines *OA* and *OB* are called the **sides** and the point *O* the **vertex** of the angle *AOB*.

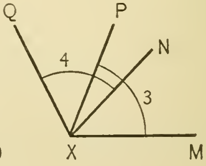


We may define an *angle* to be the figure formed by two rays, or half lines, proceeding from a common point.

**23. Notation.** If three letters are used to denote an angle, the vertex letter should be read between the others; as angle *ABC*, angle *EOF*. A single letter at the vertex denotes the largest angle at this vertex (if there be several at this point). Thus, angle *DOF* may be read " angle *O*," angle *ABC* may be read " angle *B*."







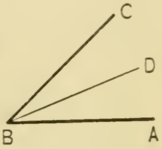
Frequently an angle is also designated by a number, or italic letter, placed within it; as angle 1, angle 2, angle *m*.

Angle *FOD* is the **sum** of angles 2 and *m*. Angle 2 is the **difference** between angles *FOD* and *m*.

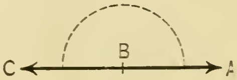
Often a curve is drawn to point out more clearly which angle is meant; as angle 2, and angle 3. An arc placed close to a number shows which angle is designated. Thus, angle *MXP* may be read "angle 3," and angle *NXQ* may be read "angle 4."

**24.** To **bisect** an angle means to divide it into two equal parts. We assume every angle has one and only one bisector.

Thus, *BD* bisects angle *ABC*, if angle *ABD* = angle *DBC*. *BD* is called the **bisector** of angle *B*.

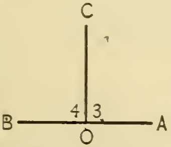


**25.** Def. A **straight angle** is an angle whose sides lie in the same straight line but extend in opposite directions; as *ABC*.

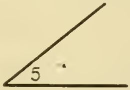


**26.** Def. A **right angle** is an angle equal to one half a straight angle.

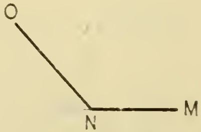
Thus, if *OC* bisects the straight angle *AOB*, angle 3 and angle 4 are right angles.



Right Angles



Acute Angle



Obtuse Angle

**27.** Def. An **acute angle** is an angle less than a right angle; as angle 5.

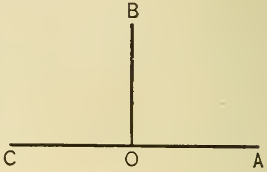
**28.** Def. An **obtuse angle** is an angle greater than a right angle, but less than a straight angle; as angle *MNO*.

**29.** Def. Acute and obtuse angles are called **oblique angles.**

**30.** Def. Two lines are **perpendicular** to each other if they meet at right angles; as *AC* and *BO*.

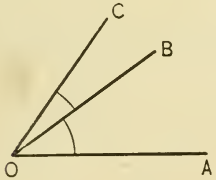
The point of meeting (*O*) is the **foot** of the perpendicular.

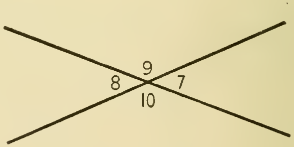
The **distance** of a point from a line is the length of the perpendicular from the point to this line; as *BO*.



**31.** An angle is **measured** by finding how many times it contains a certain unit. The usual unit is the **degree**, or one ninetieth ($\frac{1}{90}$) of a right angle. A degree is divided into sixty equal parts called **minutes**, and a minute into sixty equal parts called **seconds**. Degrees, minutes, and seconds are expressed by symbols, as 6° 50′ 12″. Read *six degrees, fifty minutes, and twelve seconds.* Other units are the right angle and the straight angle.

**32.** Def. **Adjacent angles** are two angles that have a *common vertex*, and a *common side between them;* as angles *AOB* and *BOC*.



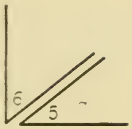


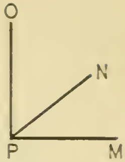
**33.** Def. Two angles are **vertical** (or **opposite**) **angles** if the sides of each are prolongations of the sides of the other back through their common vertex; as angles 7 and 8, or angles 9 and 10. $\angle 7 = \angle 8$ ; $\angle 9 = \angle 10$

**34.** Def. We can **add** two angles by placing them so that they are adjacent. The angle formed by the two sides not common is called their **sum.**

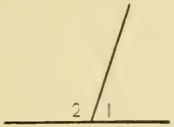
**35.** Def. Two angles are **complementary** if their sum equals a right angle.

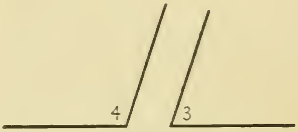
Each is then called the *complement* of the other. Angles 5 and 6, or *MPN* and *NPO*, are complementary.





**36.** Two angles are **supplementary** if their sum equals a straight angle (or two right angles).





Each angle is then called the supplement of the other. Angles 1 and 2, or angles 3 and 4, are supplementary.

---

Ex. **1.** How many degrees are in a right angle? In a straight angle? In one half a right angle?

Ex. **2.** What is the angle made by the two hands of a clock at three o'clock? At six o'clock? At two o'clock? At five o'clock?

Ex. **3.** What is the angle made by the hands of a clock at 1 P.M.? At 2:30 P.M.? At 5:30 P.M.?

Ex. **4.** Over an angle of how many degrees does a spoke of a wheel sweep when the wheel makes $\frac{1}{4}$ of a revolution? $\frac{1}{6}$ of a revolution? 2 revolutions?

Ex. **5.** How large is each angle at the center if a pie is divided into 5 equal parts? 6 equal parts?

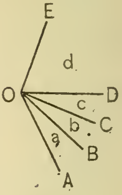
Ex. **6.** What angle is formed by lines drawn north and northeast? Towards S. and S.E.? Towards N.W. and S.W.?

Ex. **7.** Over what angle does the large hand of a watch sweep in 10 min.? 15 min.? 30 min.? 45 min.? 1 hr.?

Ex. **8.** In diagram for Ex. 9 read by three letters: $\angle a$, $\angle b$, $\angle c$, $\angle d$, $\angle(a + b)$, $\angle(b + c + d)$.

Ex. **9.** In diagrams similar to the one given here find the numerical values of the required angles:

(*a*) If $\angle a = 30°$, and $\angle b = 40°$, find $\angle AOC$.
(*b*) If $\angle b = 35°$, and $\angle c = 10°$, find $\angle BOD$.
(*c*) If $\angle b = 40°$, $\angle c = 10°$, and $\angle d = 50°$, find $\angle BOE$.
(*d*) If $\angle AOC = 60°$, and $\angle b = 40°$, find $\angle a$.
(*e*) If $\angle AOD = 90°$, $\angle a = 35°$, and $\angle c = 10°$, find $\angle b$.
(*f*) If $\angle AOE = 110°$, $\angle a = 20°$, and $\angle d = 30°$, find $\angle BOD$.
(*g*) If $\angle AOC = 60°$, and $\angle a = \angle b$, find $\angle a$.
(*h*) If $\angle AOD = 75°$, and $\angle a = \angle b = \angle c$, find $\angle c$.

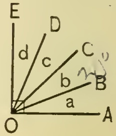


Ex. 9

Ex. **10.** In the preceding diagram, which angles are adjacent to $\angle BOC$? To $\angle COD$? To $\angle BOD$?

Ex. **11.** In diagrams similar to the one shown, if $\angle O = 90°$:

(*a*) Which angle is the complement of $\angle a$?
(*b*) Which angle is the complement of $\angle AOC$?
(*c*) Which angle is the complement of $\angle BOE$?
(*d*) If $\angle d = 20°$, find $\angle AOD$.
(*e*) If $\angle b = 20°$, and $\angle COE = 55°$, find $\angle a$.
(*f*) If $\angle AOC = 55°$, and $\angle d = 15°$, find $\angle c$.
(*g*) If $\angle a = \angle b = \angle c = \angle d$, find $\angle a$.



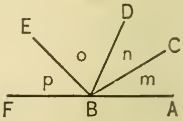
Ex. 11

Ex. **12.** How many degrees are in the complement of 30°? Of 35°? Of $\frac{2}{3}$ of a right angle? Of $n°$? Of $\frac{1}{n}$ of a right angle? Of $(10 + x)°$?

Ex. **13.** How many degrees are there in an angle that is twice its complement?

Ex. **14.** In diagrams similar to the annexed one, if *FBA* is a straight line,

(*a*) Which angle is the supplement of $\angle p$?
(*b*) Which angle is the supplement of $\angle DBF$?
(*c*) Which angle is the supplement of $\angle ABE$?
(*d*) If $\angle p = 40°$, find $\angle ABE$.
(*e*) If $\angle m = 30°$, and $\angle p = 35°$, find $\angle CBE$.
(*f*) If $\angle DBF = 100°$, and $\angle m = \angle n$, find $\angle m$.
(*g*) If $\angle p = 30°$, and $\angle m = \angle n = \angle o$, find $\angle o$.
* (*h*) If $\angle FBC = 140°$, and $\angle ABD = 80°$, find $\angle n$.
* (*i*) If $\angle ABD = 80°$, $\angle n = 35°$, and $\angle CBE = 85°$, find $\angle p$.



Ex. 14

Ex. **15.** How many degrees are in the supplement of 20°? Of 140°? Of $\frac{3}{4}$ of a straight angle? Of *n* degrees? Of $(50 - 3x)°$?

*Exercises denoted by (*) are difficult.

**Ex. 16.** How many degrees are in an angle which is three times its supplement?

**Ex. 17.** What kind of angle is less than its supplement? Equal to its supplement? Greater than its supplement?

**Ex. 18.** Write in algebraic symbols:
(*a*) The complement of $n^\circ$. (*b*) Three times the complement of $x^\circ$.
(*c*) The supplement of $(2x)^\circ$. (*d*) Six times the supplement of $n^\circ$.

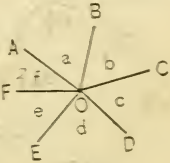
**Ex. 19.** In diagrams similar to the annexed one find the numerical values of the required angles:

(*a*) If $\angle a = 80^\circ$, $\angle b = 50^\circ$, $\angle c = 60^\circ$, $\angle d = 90^\circ$, and $\angle e = 50^\circ$, find $\angle f$.

(*b*) If $\angle a = \angle b = \angle c = \angle d = \angle e = \angle f$, find $\angle f$.

* (*c*) If $\angle AOC = 130^\circ$, $\angle b = 50^\circ$, $\angle BOD = 110^\circ$, and $\angle DOF = 140^\circ$, find $\angle f$.

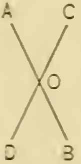
(*d*) If $\angle d = 90^\circ$, and $\angle c = \angle b = \angle a = \angle f = \angle e$, find $\angle a$.



Ex. 19

**Ex. 20.** If two lines, $AB$ and $CD$, intersect in $O$, making $\angle AOC = 60^\circ$, find the other angles.

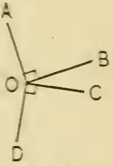
**Ex. 21.** If $\angle AOC = m$ degrees, how many degrees are in $\angle DOB$? In $\angle BOC$?



Exs. 20, 21

**Ex. 22.** If $\angle AOB = \angle COD = 90^\circ$, find $\angle AOD$ (*a*) if $\angle BOC = 60^\circ$. (*b*) if $\angle BOC = m^\circ$.

**Ex. 23.** What relation exists between the angles $BOC$ and $AOD$?

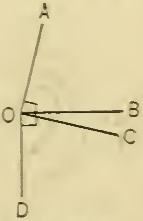


Exs. 22, 23

**Ex. 24.** If $AO$ is perpendicular to $CO$, and $BO$ is perpendicular to $DO$, find $\angle AOD$, (*a*) if $\angle COB = 40^\circ$. (*b*) if $\angle COB = m^\circ$.

**Ex. 25.** What relation exists between $\angle AOD$ and $\angle BOC$?

* **Ex. 26.** If $\angle AOC = \angle BOD = 90^\circ$, and $\angle AOD = 3\angle BOC$, find $\angle BOC$.



Exs. 24–26

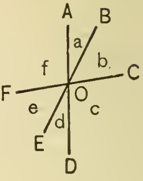
Ex. **27**. Three lines meet in $O$ forming the six angles: $a$, $b$, $c$, $d$, $e$, and $f$.

($a$) If $\angle a = 20°$, and $\angle b = 60°$, find $\angle c$.

* ($b$) If $\angle a = 15°$, and $\angle c = 95°$, find $\angle e$.

* ($c$) If $\angle f = 100°$, and $\angle d = 20°$, find $\angle b$.

* ($d$) If $\angle AOC = 85°$, and $\angle BOD = 155°$, find $\angle e$.



Ex. 27

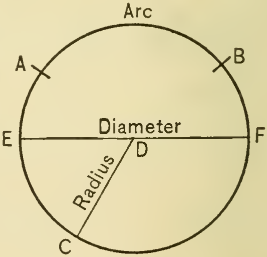
Ex. **28**. Find the angle formed by the bisectors of the supplementary adjacent angles $AOB$ and $BOC$, ($a$) if $\angle AOB = 40°$, ($b$) if $\angle AOB = 60°$, ($c$) if $\angle OAB = m°$.

Ex. **29**. *What is the angle formed by the bisectors of any two supplementary adjacent angles?*

Ex. **30**. Find the angle formed by the bisectors of the complementary adjacent angles $AOB$ and $BOC$: ($a$) if $\angle AOB = 20°$, ($b$) if $\angle AOB = 30°$, ($c$) if $\angle AOB = m°$.

Ex. **31**. What is the angle formed by the bisectors of any two complementary adjacent angles?

**37**. Def. A **circle** is a plane closed curve, having all of its points equally distant from a fixed point ($D$); as $ABC$.



The **center** ($D$) is the fixed point. A **radius** is any straight line from the center to a point in the circle; as $DC$. An **arc** is any portion of the circle; as $AB$. The length of the circle is called the **circumference**.

The term *circumference* is frequently used to denote the curved line, and the term *circle* to denote the area. But modern usage is against this terminology.

**38. Instruments used in geometry.** Only two instruments are permitted to be used in plane geometry, viz., the *compasses* and the *straight edge*.

The straight edge is employed for the drawing of straight lines, the compasses for the drawing of circles or arcs and the transfer of lines of definite length (segments) from one position to another.[1]

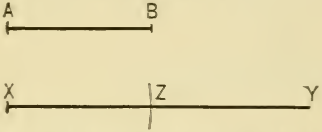
---

[1] A line in a plane is said to be closed if it separates a finite portion of the plane from the remaining portion.

## EXERCISES IN GEOMETRIC DRAWING

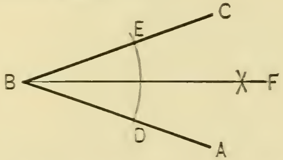
**39.** The following exercises are designed to familiarize the student with the use of the instruments and to impress upon him the meaning of the fundamental notions of geometry. They form, however, no logical part of plane geometry, and may be omitted without affecting the course. They are based upon three constructions, for which no proofs are given here. (Compare §§ 83 and 86.)

I. *To lay off on XY a line equal to AB.*



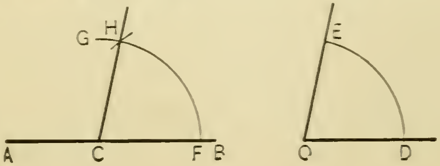
From $X$ as a center with a radius equal to $AB$ draw an arc intersecting $XY$ in $Z$; $XZ$ is the required line. (§ 38.)

II. *To bisect a given angle ABC.*



From $B$ as a center with any radius draw an arc meeting $AB$ in $D$, and $AC$ in $E$. From $D$ and $E$ as centers with a radius sufficiently large draw two arcs intersecting in $F$. Then the line $BF$ is the required bisector.

III. *From a point C in a given straight line AB, to draw a line to make an angle with AB equal to angle O.*



From $O$ as a center, with any radius, describe an arc cutting the sides of $\angle O$ in $D$ and $E$. From $C$ as a center with the same radius describe arc $FG$, intersecting $AB$ in $F$. From $F$ as a center with a radius equal to $DE$,[1] draw an arc intersecting $FG$ in $H$. Draw $CH$. Then $\angle BCH$ is the required angle.

Ex. **1.** Draw two points $A$ and $B$ (indicated by small dots or crosses) and construct a line through $A$ and $B$.

Ex. **2.** Draw two points $A$ and $B$, and construct a circle from $A$ as a center with a radius equal to $AB$.

Ex. **3.** Draw three points $A$, $B$, and $C$, and construct a straight line through every pair of points.

Ex. **4.** Draw two lines of definite length, $AB$ and $CD$, $AB$ being the longer line, and construct lines equal to:

(*a*) $AB + CD$
(*b*) $AB - CD$
(*c*) $2(AB)$
(*d*) $AB + 2(CD)$
(*e*) $3(AB) - 2(CD)$

Ex. **5.** Draw an acute angle and bisect it.

Ex. **6.** Draw an obtuse angle and bisect it.

Ex. **7.** *Construct a right angle and bisect it.*

Ex. **8.** Draw a straight angle and bisect it.

Ex. **9.** Construct the sum of two angles.

Ex. **10.** At a given point $C$ in $AB$ draw a perpendicular to $AB$.

Ex. **11.** Divide a given angle into 4 equal parts.

Ex. **12.** Divide a given angle into 8 equal parts.

Ex. **13.** Construct an angle of: $90°$; of $45°$.

Ex. **14.** Construct an angle of: $22° 30'$; of $135°$.

Ex. **15.** Construct an angle of: $270°$; of $67° 30'$.

Ex. **16.** *Construct the supplement of a given angle $A$.*

Ex. **17.** Construct one half the supplement of a given angle $A$.

Ex. **18.** *Construct the complement of a given acute angle.*

Ex. **19.** Construct one half the complement of a given acute angle.

---

[1] Two letters, *e.g.* $DE$, used as above, denote a straight line.

Ex. **20**. Construct the supplement of the complement of a given acute angle.

Ex. **21**. Construct the complement of the supplement of a given obtuse angle.

Ex. **22**. Draw two angles $A$ and $B$, $A$ being the greater one, and construct an angle equal to:

(*a*) $A + B$

(*b*) $2\,A$

(*c*) $180° - A$

(*d*) $90° + A$

(*e*) $A - B$

(*f*) $\frac{A}{2}$

(*g*) $\frac{A}{2} + B$

(*h*) $\frac{A + B}{2}$

(*i*) $\frac{A}{2} + \frac{B}{2}$

(*j*) $\frac{A}{2} - \frac{B}{2}$

(*k*) $90 + \frac{B}{2}$

(*l*) The complement of $\frac{B}{4}$

## GENERAL TERMS

✓ **40.** Def. A **theorem** is a statement the truth of which is to be demonstrated (or proved).

Thus the statement: "*If two sides of a triangle are equal, the angles opposite are equal,*" is a theorem. The conditional part of the statement is called the *hypothesis*, the assertion that is to be proved is called the *conclusion* Thus in the above example, the hypothesis is: "*If two sides of a triangle are equal,*" and the conclusion is: "*the angles opposite are equal.*" The hypothesis is, however, not always explicitly stated, but is sometimes merely implied.

**41.** Def. A **problem**, in general, is a question to be solved.

The problems of geometry are solved either by constructions or by computations. Thus the problem: "To construct an angle of 45°" (Ex. 13, p. 12) is a *problem of construction.* "To compute the angle formed by the bisectors of a pair of complementary adjacent angles" (Ex. 30, p. 10) is a *problem of computation.*

**42.** Def. A **proposition** is a general term for either a theorem or a problem.

**43.** Def. An **axiom** is a statement whose truth is assumed.

Thus, the statement: " If equals be added to equals, the sums are equal," is an axiom.

**44.** Def. A **postulate** is a purely geometric axiom.

Formerly elementary textbooks restricted the term *postulate* to ‘ geometric constructions whose possibility is admitted without further demonstration," but modern usage favors the more general definition given above.

**45.** Def. A **corollary** is a theorem easily derived from another theorem.

## AXIOMS AND POSTULATES

1. *Things equal to the same thing, or to equal things, are equal to each other.*

2. *If equals are added to equals, the sums are equal.*

3. *If equals are subtracted from equals, the remainders are equal.*

4. *If equals are added to unequals, the sums are unequal in the same sense.*

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

6. *If unequals are subtracted from equals, the remainders are unequal in the opposite sense.*

7. *If equals are multiplied by equals, the products are equal.* (Important special case: *doubles of equals are equal.*)

8. *If equals are divided by equals, the quotients are equal.* (Important special case: *halves of equals are equal.*)

9. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

10. *The whole is equal to the sum of all its parts.*

11. *The whole is greater than any of its parts.*

12. *A quantity may be substituted for an equal one in an equation or in an inequality.* (Briefly called " *substitution.*")

13. *One straight line and only one can be drawn through two points* (*or two points determine a line*).

14. *A straight line is the shortest line between two points.*

15. *All straight angles are equal.*

16. *Two intersecting straight lines cannot both be parallel to a third straight line.*

17. *A straight line may be produced to any required length.*

18. *A circle may be drawn having any point as center and any line as radius.*

19. *A geometric figure may be moved from one position to another without change of form or size.*

The first twelve axioms are general axioms, the rest are geometric axioms or postulates.

A straight line is fully determined, given a point on this line and given the angle it makes with a given line.

From the thirteenth axiom these truths obviously follow:

(*a*) Two straight lines can intersect in only one point.

(*b*) Two lines of unlimited length, coinciding in part, coincide throughout.

---

Ex. **1.** Indicate the hypothesis and the conclusion of each of the following statements:

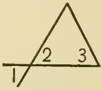
(*a*) If iron is heated, it expands.

(*b*) If two angles of a triangle are equal, the opposite sides are equal.

(*c*) Two triangles are congruent if the sides of the one are respectively equal to the sides of the other.

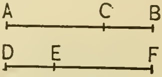
(*d*) Vertical angles are equal.

Ex. **2.** If $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$, for what reason[1] does $\angle 1 = \angle 3$?



Ex. 2

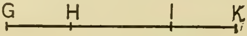
Ex. **3.** If $AB = DF$, and $CB = DE$, for what reason does $AC = EF$?



Ex. 3

Ex. **4.** If $GH = IK$, for what reason does $GI = HK$?

Ex. **5.** If $GI = HK$, for what reason does $GH = IK$?



Exs. 4, 5

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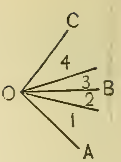
[1] The reasons requested in this and the following exercises are axioms.

**Ex. 6.** If $OB$ bisects $\angle O$, and $\angle 1 = \angle 4$, why does $\angle 2 = \angle 3$?

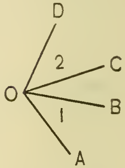
**Ex. 7.** If $\angle 1 = \angle 4$, and $\angle 3 = \angle 2$, why does $OB$ bisect $\angle O$?



Exs. 6, 7

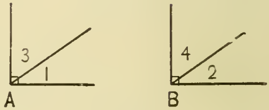
**Ex. 8.** If $\angle 1 = \angle 2$, why does $\angle AOC = \angle BOD$?

**Ex. 9.** If $\angle AOC = \angle BOD$, why does $\angle 1 = \angle 2$?



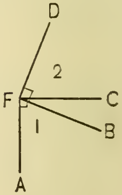
Exs. 8, 9

**Ex. 10.** If $\angle A = \angle B$ and $\angle 1 = \angle 2$, why does $\angle 3 = \angle 4$?



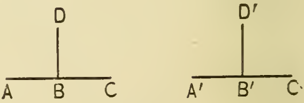
Ex. 10

**Ex. 11.** If $\angle AFC = 90°$, and $\angle BFD = 90°$, why does $\angle 1 = \angle 2$?



Ex. 11

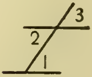
**Ex. 12.** If st. $\angle ABC =$ st. $\angle A'B'C'$, why does rt. $\angle DBC =$ rt. $\angle D'B'C'$?



Ex. 12

**Ex. 13.** If $\angle 1 + \angle b + \angle 2 = 180°$, $\angle 1 = \angle a$, and $\angle 2 = \angle c$, why does $\angle a + \angle b + \angle c = 180°$?

**Ex. 14.** If $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$, why does $\angle 1 = \angle 3$?



Ex. 14

# BOOK I

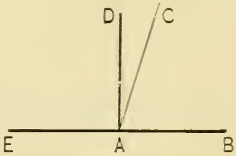
## LINES AND RECTILINEAR FIGURES

### PRELIMINARY THEOREMS

**46.** *All right angles are equal.*

For all straight angles are equal (Ax. 15), and halves of equals are equal (Ax. 8).

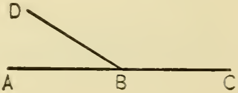
**47.** *At a given point in a given line one and only one perpendicular can be drawn to the line.*



For if two perpendiculars, *AD* and *AC*, could be drawn at *A*, we should have two unequal right angles *BAD* and *BAC*, which is impossible (46).

**48.** *Complements of the same angle or of equal angles are equal.* (Ax. 3.)

**49.** *Supplements of the same angle or of equal angles are equal.* (Ax. 3.)

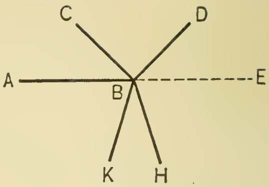


**50.** *If two adjacent angles have their exterior sides in a straight line, they are supplementary.* (Ax. 10.)

**51.** *If two adjacent angles are supplementary, their exterior sides are in the same straight line.*

Their exterior sides form a straight angle (36), and hence lie in a straight line (25).

**52.** *The sum of all the angles about a point in a plane is equal to two straight angles.*



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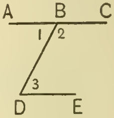
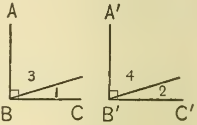
Ex. **1.** If $ABC$ is a straight line, and $\angle 3$ is the supplement of $\angle 2$, why is $\angle 1 = \angle 3$?



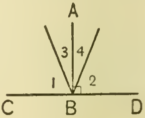
Ex. 1

Ex. **2.** If $AB \perp BC$, $A'B' \perp B'C'$, and $\angle 3 = \angle 4$, prove $\angle 1 = \angle 2$.



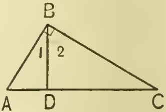
Ex. 2

Ex. **3.** If $AB \perp CD$ and $\angle 1 = \angle 2$, prove $\angle 3 = \angle 4$.



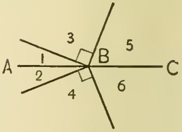
Ex. 3

Ex. **4.** If $\angle ABC$ is a right angle and $\angle A$ is the complement of $\angle 1$, prove $\angle A = \angle 2$.



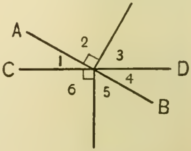
Ex. 4

Ex. **5.** If $ABC$ is a st. $\angle$, $\angle 1 = \angle 2$, and $\angle$s 3 and 4 are rt. $\angle$s, prove that $\angle 5 = \angle 6$.



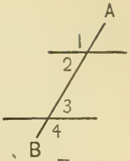
Ex. 5

Ex. **6.** If $AB$ and $CD$ are straight lines and $\angle$s 2 and 6 are right angles, prove that $\angle 3 = \angle 5$.



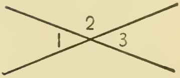
Ex. 6

Ex. **7.** If $AB$ is a straight line and $\angle 2 = \angle 3$, prove that $\angle 1 = \angle 4$.



Ex. 7

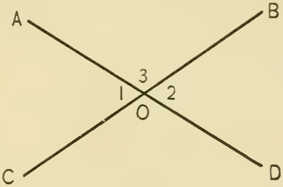
Ex. **8.** If $\angle 1$ is the supplement of $\angle 2$ and $\angle 3$ is the supplement of $\angle 2$, why is $\angle 1 = \angle 3$?



Ex. 8

### PROPOSITION I. THEOREM

✓ **53.** *Vertical angles are equal.*



**Given** the vertical $\measuredangle$ 1 and 2.

**To prove** $\angle 1 = \angle 2$.

**Proof**

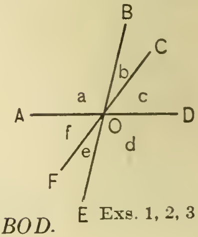
| STATEMENTS | REASONS |
|---|---|
| **1.** $AOD$ and $COB$ are straight lines. | **1.** By hyp. $\angle 1$ and $\angle 2$ are vertical $\measuredangle$. |
| **2.** $\angle 1$ is the sup. of $\angle 3$. | **2.** If two adj. $\measuredangle$ have their ext. sides in a st. line, they are sup. |
| **3.** $\angle 2$ is the sup. of $\angle 3$. | **3.** Same reason as 2. |
| **4.** $\therefore \angle 1 = \angle 2$. | **4.** Supplements of the same $\angle$ are equal. |

**54.** Every proof consists of a number of statements, each of which is supported by a definite reason. The only admissible reasons are: a previously proved proposition; an axiom; a definition; or the hypothesis.

[Practical applications, pages 281–286.]

Ex. **1.** Three straight lines, $AD$, $BE$, and $CF$, meet in $O$, forming the six angles: $a$, $b$, $c$, $d$, $e$, and $f$.

(*a*) If $\angle b = 20^\circ$, and $\angle c = 60^\circ$, find $\angle AOE$.
(*b*) If $\angle FOB = 130^\circ$, and $\angle c = 40^\circ$, find $\angle a$.
(*c*) If $\angle FOD = 140^\circ$, and $\angle b = 60^\circ$, find $\angle a$.
(*d*) If $\angle f = 60^\circ$, and $\angle b = 25^\circ$, find $\angle d$.
(*e*) If $\measuredangle b$ and $f$ are complementary, find $\angle d$.
(*f*) If $\angle f = \angle b$, and $\angle d = 100^\circ$, find $\angle b$.
(*g*) If $\angle a = 2\angle c$, and $\angle e = 60^\circ$, find $\angle c$.
(*h*) If $\angle AOC = 140^\circ$, and $\angle COE = 120^\circ$, find $\angle BOD$.



Exs. 1, 2, 3

Ex. **2.** In the same diagram prove that

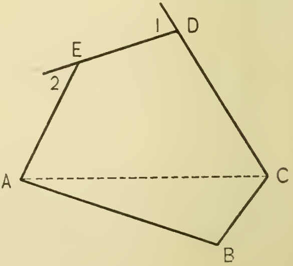
(*a*) $\angle b + \angle c = \angle AOE$.
(*b*) $\angle FOB - \angle c = \angle a$.
(*c*) $\angle FOB - \angle f = \angle d$.
(*d*) $\angle f + \angle b + \angle d = 180^\circ$.
(*e*) $\angle AOC + \angle BOD + \angle COE = 360^\circ$.
(*f*) $\angle AOC + \angle COE - \angle EOA = 2\angle a$.
(*g*) If $\angle f = \angle e$, then $\angle b = \angle c$.

[1]* Ex. **3.** In the same diagram

(*a*) If $\angle AOC = 150^\circ$, and $\angle COE = 130^\circ$, find $\angle a$.
(*b*) If $\angle FOB = 140^\circ$, and $\angle AOC = 125^\circ$, find $\angle d$.
(*c*) If $\angle AOE + \angle BOC = 140^\circ$, and $\angle c = 40^\circ$, find $\angle e$.
(*d*) If $\angle FOB = \angle FOD$, prove that $\angle f = \angle e$.

**55.** Def. A **polygon** is a closed broken line. The lines are called the **sides**. The **perimeter** of a polygon is the sum of all its sides.

The angles formed by the consecutive sides are the **angles** of the polygon, and their vertices are the **vertices** of the polygon. An **exterior angle** is formed by a side and the prolongation of a consecutive one. A **diagonal** is a straight line joining two vertices not on a same side of the polygon.



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[1] Exercises denoted by (*) are difficult and may be omitted in a first reading.

Thus $ABCDE$ is a polygon of five sides, $AB$ and $BC$ are sides, $\angle A$ is an angle, point $B$ is a vertex, and $AC$ is a diagonal of the polygon. $\angle 1$ and $\angle 2$ are exterior angles of the polygon.

**56.** Def. A **quadrilateral** is a polygon of four sides.

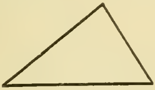
## TRIANGLES — PART I

**57.** A **triangle** is a polygon of three sides.

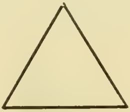
**58. Triangles classified as to sides.** A **scalene** triangle is a triangle no two sides of which are equal.

An **isosceles** triangle is a triangle having two equal sides.

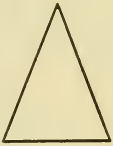
An **equilateral** triangle is a triangle having all of its sides equal. It is a special kind of isosceles triangle.



Scalene



Equilateral



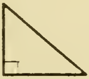
Isosceles

**59. Triangles classified as to angles.** A **right** triangle is a triangle in which one angle is *a right angle.*

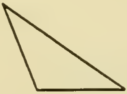
An **obtuse** triangle is a triangle in which one angle is *an obtuse angle.*

An **acute** triangle is a triangle in which *all angles* are *acute.*

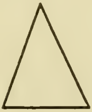
An **equiangular** triangle is a triangle in which *all angles* are equal.



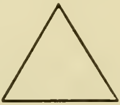
Right



Obtuse



Acute



Equiangular

**60.** Def. The **base** of a triangle is the side on which the figure appears to stand. Any side may be considered as a base.

The two equal sides of an isosceles triangle are sometimes called the *arms* (or *legs*); the third side is called the *base*.

The angles opposite the arms are *the base angles* of an isosceles triangle.

**61.** Def. The **vertex angle of a triangle** is the angle opposite the base.

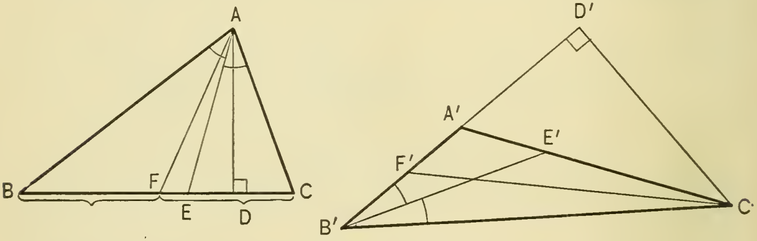
The vertex of this angle is sometimes called "the vertex of the triangle."

**62.** Def. The **hypotenuse** of a right triangle is the side opposite the right angle.

The sides including the right angle are sometimes called the *arms* (*legs*) of the right triangle.

**63.** Def. An **altitude** of a triangle is the perpendicular from any vertex to the opposite side (produced if necessary).

Thus, $AD$ and $C'D'$ are altitudes. Every triangle has three altitudes.



**64.** Def. A **median** of a triangle is the line drawn from any vertex to the mid-point of the opposite side.

Thus, $AF$ and $C'F'$ are medians. Every triangle has three medians.

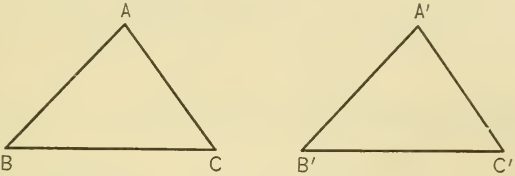
**65.** Def. An **angle-bisector** of a triangle is the line drawn from any vertex bisecting the angle at that vertex and limited by the side opposite this vertex.

Thus, $AE$ and $B'E'$ are angle-bisectors. Every triangle has three angle-bisectors.

Note. Students who are familiar with the exercises on geometric drawing (39) should construct exactly a number of the diagrams defined above, as right ◬, obtuse ◬, medians, angle-bisectors, equilateral ◬, etc.

### PROPOSITION II. THEOREM

**66.** *Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other* $(a.s.a. = a.s.a.)$.



**Given** $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $\angle A = \angle A'$, and $\angle B = \angle B'$.

**To prove** $\triangle ABC \cong \triangle A'B'C'$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that $AB$ shall coincide with its equal $A'B'$. | 1. Ax. 19.<br>$AB = A'B'$, by hyp. |
| 2. Then $BC$ will fall along $B'C'$. | 2. $\angle B = \angle B'$, by hyp. |
| 3. $AC$ will fall along $A'C'$. | 3. $\angle A = \angle A'$, by hyp. |
| 4. $C$ will fall upon $C'$. | 4. Two st. lines can intersect in only one point. |
| 5. $\therefore \triangle ABC \cong \triangle A'B'C'$. | 5. Figures are congruent if they can be made to coincide. |

**67.** This method of proof (**superposition**) is employed in fundamental propositions only. The student should place some parts upon each other whose equality is known, and, by successive steps, trace the position of the rest of the figure.

NOTE. In order to facilitate the citing of propositions, the following abbreviation is suggested for the above proposition: a. s. a. = a. s. a.

**68.** Def. Polygons are **mutually equiangular** if their angles are respectively equal, and **mutually equilateral** if their sides are respectively equal.

If two polygons are mutually equiangular, lines or angles similarly situated are called **corresponding** (or **homologous**) **lines or angles.** Thus $AB$ and $A'B'$ (Prop. II) are corresponding sides; $C$ and $C'$, corresponding angles, the medians drawn from $A$ and $A'$ respectively, corresponding medians; etc.

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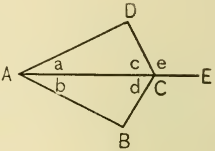


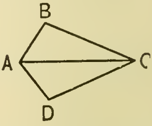
Exs. 1, 2

Ex. **1.** **Hyp.** $\angle a = 30^\circ$, $\angle b = 30^\circ$, $\angle c = 60^\circ$, $\angle d = 60^\circ$.

**To prove** $\triangle ABC \cong \triangle ADC$.

Ex. **2.** **Hyp.** $\angle a = 40^\circ$, $\angle b = 40^\circ$, $\angle e = 130^\circ$, $\angle d = 50^\circ$, and $ACE$ is a straight line.

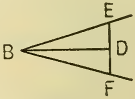
**To prove** $\triangle ABC \cong \triangle ADC$.



Ex. 3

Ex. **3.** **Hyp.** In quadrilateral $ABCD$, $AC$ bisects angle $A$, and $AC$ bisects $\angle C$.

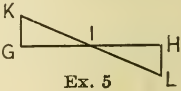
**To prove** $\triangle ABC \cong \triangle ADC$.



Ex. 4

Ex. **4.** **Hyp.** $BD$ bisects $\angle B$, $EF \perp BD$.

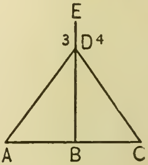
**To prove** $\triangle EBD \cong \triangle FBD$.



Ex. 5

Ex. **5.** **Hyp.** $I$ is the mid-point of $GH$, $KG \perp GH$, $HL \perp GH$, $KL$ is a straight line.

**To prove** $\triangle KGI \cong \triangle HLI$.



Exs. 6, 7

Ex. **6.** **Hyp.** $DB \perp AC$, $\angle 3 = \angle 4$, and $BDE$ is a straight line.

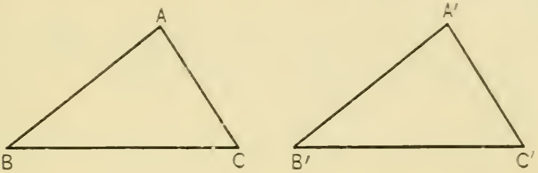
**To prove** $\triangle ABD \cong \triangle CBD$.

Ex. **7.** **Hyp.** $DB \perp AC$, $\angle ADB$ is the sup. of $\angle 4$, and $BDE$ is a st. line.

**To prove** $\triangle ABD \cong \triangle CBD$.

## Proposition III. Theorem

**69.** *Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other* (s. a. s. = s. a. s.)



**Given** $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$ and $\angle B = \angle B'$.

**To prove** $\triangle ABC \cong \triangle A'B'C'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Place $\triangle ABC$ upon $\triangle A'B'C'$ so that $BC$ shall coincide with its equal $B'C'$. | **1.** A geom. fig. may be . . (Ax. 19.)<br>$BC = B'C'$, by hyp. |
| **2.** $BA$ will fall along $B'A'$. | **2.** $\angle B = \angle B'$, by hyp. |
| **3.** The point $A$ will fall upon the point $A'$. | **3.** $AB = A'B'$, by hyp. |
| **4.** $AC$ will coincide with $A'C'$. | **4.** One st. line and only one can be drawn through two points. |
| **5.** $\therefore \triangle ABC \cong \triangle A'B'C'$. | **5.** Figures are congruent if they can be made to coincide. |

Ex. 1. **Hyp.** Two lines $AB$ and $CD$ bisect each other in $E$.

**To prove** $\triangle ADE \cong \triangle CEB$.[1]

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[1] The following set of exercises is of fundamental importance. No student should go on with the work until he is able to discover the proofs of all the simpler theorems given below. For additional work of this kind see Schultze's Teaching of Mathematics.

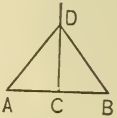
**70.** Method I. **The equality of lines and of angles is usually proved by means of congruent triangles.**

Note 1. The equal lines will be the corresponding sides in the congruent triangles. Likewise for the equal angles.

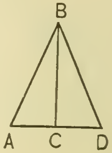
Note 2. The student is advised to draw freehand each figure accompanying each exercise.

Ex. **2.** **Hyp.** $DC$ is the perpendicular bisector of $AB$, (*i.e.* $AC = CB$ and $DC \perp AB$.)

**To prove** $AD = BD$.

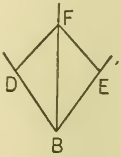


Ex. 2



Ex. 3

Ex. **3.** The bisector of the vertex angle of an isosceles triangle bisects the base.



Ex. 4

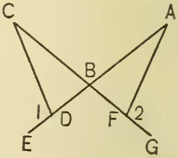
Ex. **4.** **Hyp.** $BF$ bisects $\angle DBE$.

$BD = BE$.

**To prove** $\angle FDB = \angle FEB$.

Ex. **5.** **Hyp.** $BD = BF$, $BC = BA$, and $CG$ and $AE$ are st. lines.

**To prove** $CD = AF$.



Exs. 5–8

Ex. **6.** **Hyp.** $BD = BF$, $CD \perp EA$, $AF \perp CG$, and $CG$ and $AE$ are st. lines.

**To prove** $\angle C = \angle A$.

Ex. **7.** **Hyp.** $BD = BF$, $\angle 1 = \angle 2$, and $CG$ and $AE$ are st. lines.

**To prove** $CB = AB$.

Ex. **8.** **Hyp.** $CF = AD$, $DB = FB$, and $CG$ and $AE$ are st. lines.

**To prove** $\angle C = \angle A$.

Ex. **9.** **Hyp.** $AD = DC$, $\angle 5 = \angle 6$, and $BE$ is a st. line.

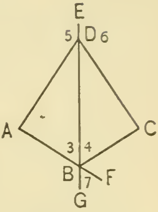
**To prove** $\angle A = \angle C$.

Ex. **10.** **Hyp.** $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and $BE$ is a st. line.

**To prove** $AD = DC$.

Ex. **11.** **Hyp.** $AB = BC$, $\angle 4 = \angle 7$, and $AF$ and $EG$ are st. lines.

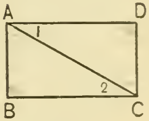
**To prove** $AD = DC$.



Exs. 9–11

Ex. **12.** **Hyp.** $AB \perp AD$, $DC \perp BC$ and $\angle 1 = \angle 2$.

**To prove** $\angle B = \angle D$.



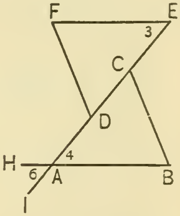
Ex. 12

Ex. **13.** **Hyp.** $AD = CE$, $AB = FE$, $\angle 3 = \angle 4$, and $AE$ is a st. line.

**To prove** $\angle B = \angle F$.

Ex. **14.** **Hyp.** $AD = CE$, $AB = FE$, $\angle 3 = \angle 6$, and $EI$ and $HB$ are st. lines.

**To prove** $BC = FD$.



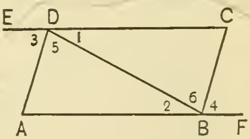
Exs. 13, 14

Ex. **15.** **Hyp.** $\angle EDB = \angle FBD$, $\angle 5 = \angle 6$, and $CE$ and $AF$ are st. lines.

**To prove** $AD = BC$.

Ex. **16.** **Hyp.** $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and $EC$ and $AF$ are st. lines.

**To prove** $AB = DC$.



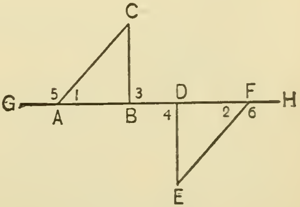
Exs. 15, 16

Ex. **17.** **Hyp.** $AD = BF$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and $GH$ is a st. line.

**To prove** $\angle C = \angle E$.

Ex. **18.** **Hyp.** $AD = BF$, $\angle 5 = \angle 6$, $AC = FE$, and $GH$ is a st. line.

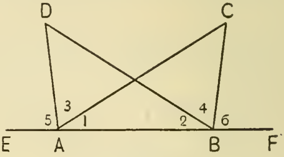
**To prove** $BC = DE$.



Exs. 17, 18

Ex. **19.** **Hyp.** $AD \perp EF, CB \perp EF, AD = BC$, and $EF$ is a st. line.

**To prove** $\angle C = \angle D$.



Exs. 19–22

Ex. **20.** **Hyp.** $AD = CB, \angle 5 = \angle 6$, and $EF$ is a st. line.

**To prove** $\angle 1 = \angle 2$.

Ex. **21.** **Hyp.** $\angle 5 = \angle 6, \angle 1 = \angle 2$, and $EF$ is a st. line.

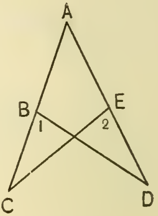
**To prove** $AC = DB$.

Ex. **22.** **Hyp.** $\angle 5 = \angle 6, \angle 3 = \angle 4$, and $EF$ is a st. line.

**To prove** $AD = BC$.

Ex. **23.** **Hyp.** $AB = AE, BC = ED$, and $AC$ and $AD$ are st. lines.

**To prove** $CE = BD$.



Exs. 23–25

Ex. **24.** **Hyp.** $AB = AE, \angle 1 = \angle 2$, and $AC$ and $AD$ are st. lines.

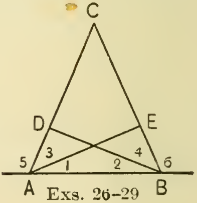
**To prove** $CE = BD$.

[1] Ex. **25.** **Hyp.** $AC = AD$, and $BC = ED$.

**To prove** $\angle C = \angle D$.

Ex. **26.** **Hyp.** $\angle 1 = \angle 2, \angle 3 = \angle 4$, and $AE$ and $BD$ are st. lines.

**To prove** $AD = BE$.



Exs. 26–29

Ex. **27.** **Hyp.** $\angle DAB = \angle EBA$, and $\angle 3 = \angle 4$.

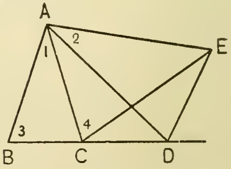
**To prove** $AD = BE$.

Ex. **28.** **Hyp.** $\angle 5 = \angle 6, \angle 1 = \angle 2$, and $AE$ and $DB$ are st. lines.

**To prove** $AD = BE$.

Ex. **29.** **Hyp.** $\angle 5 = \angle 6, AD = BE$, and $AE$ and $BD$ are st. lines.

**To prove** $\angle 1 = \angle 2$.



Exs. 30, 31

Ex. **30.** **Hyp.** $BCD$ is a st. line, $AB = AC$, $AD = AE$, and $\angle 1 = \angle 2$.

**To prove** $BD = CE$.

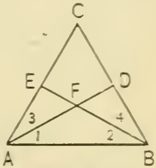
Ex. **31.** **Hyp.** $AB = AC, \angle 1 = \angle 2, \angle 3 = \angle 4$, and $AD$ is a st. line.

**To prove** $AD = AE$.

---

[1] If a line is denoted by two letters only, as $AC$ or $AD$, it signifies that it is a straight line.

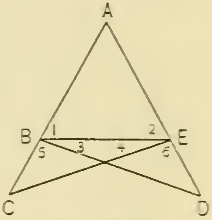
Note. The following exercises are more difficult because the two parts whose equality is to be demonstrated may be considered corresponding parts of two different pairs of triangles, and the student has to determine by trial which pair must be used.



Exs. 32, 33

Ex. **32.** **Hyp.** $\angle A = \angle B$, and $\angle 3 = \angle 4$.
To prove $AD = BE$.

Ex. **33.** **Hyp.** $AC = BC$, and $\angle 3 = \angle 4$.
To prove $AD = BE$.



Exs. 34–36

Ex. **34.** **Hyp.** $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.
To prove $EC = BD$.

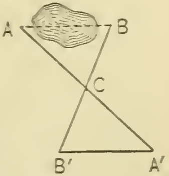
* Ex. **35.** **Hyp.** $\angle 1 = \angle 2$, $DB \perp AC$, and $CE \perp AD$.
To prove $\angle C = \angle D$.

Ex. **36.** **Hyp.** $AB = AE$, $\angle 5 = \angle 6$.
To prove $EC = BD$.

Ex. **37.** *If two angles of a triangle are equal, the corresponding angle-bisectors are equal.*

Ex. **38.** *The medians drawn to the arms of an isosceles triangle are equal.*

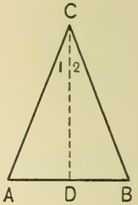
Ex. **39.** If two exterior angles of a triangle are equal, the angle-bisectors of the adjacent interior angles are equal.



Ex. 40

Ex. **40.** Wishing to determine the distance across a pond, $AB$, we place a stake at a convenient station $C$, and by sighting from $A$, we locate $A'$ in the prolongation of $AC$, and make $CA' = AC$. Similarly we produce $BC$ to $B'$ so that $CB' = BC$. Which line must we measure to obtain $AB$? Prove your statement.

### PROPOSITION IV. THEOREM



**71.** *The base angles of an isosceles triangle are equal.*

**Given** $\triangle ABC$ with $AC = BC$.
**To prove** $\angle A = \angle B$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw the bisector of $\angle C$ and extend it to meet $AB$ at $D$. | 1. Every $\angle$ has a bisector.<br>A line may be extended. |
| 2. In $\triangle$s $ACD$ and $BCD$, $AC = CB$. | 2. Hyp. |
| 3. $\angle 1 = \angle 2$. | 3. By construction. |
| 4. $CD = CD$. | 4. Iden. |
| 5. $\triangle ACD \cong \triangle BCD$. | 5. Two $\triangle$s are $\cong$ if two sides and the included angle of one are equal respectively to two sides and the included angle in the other. |
| 6. $\therefore \angle A = \angle B$. | 6. Corresponding angles of congruent $\triangle$s are equal. |

**72.** COR. 1. *The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.*

**73.** COR. 2. *An equilateral triangle is equiangular.*

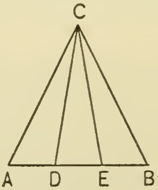
---

Ex. **1.** If $AC = BC$, and $AE = DB$, then $CD = CE$.

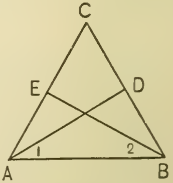
Ex. **2.** If $AC = BC$, and $\angle DCA = \angle ECB$, then $CD = CE$.

Ex. **3.** If $AC = BC$, and $\angle 1 = \angle 2$, then $AE = BD$.

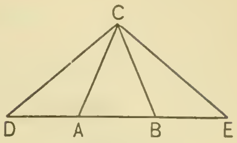
Ex. **4.** If $AC = BC$, and $AD$ and $BE$ are angle-bisectors, then $AD = BE$.



Exs. 1, 2



Exs. 3, 4

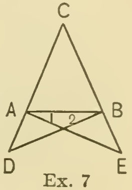


Exs. 5, 6

Ex. **5**. If $AC = BC$, $AD = BE$, and $DE$ is a straight line, then $DEC$ is an isosceles triangle.

Ex. **6**. If $AC = BC$, $\angle ACD = \angle BCE$, and $DE$ is a straight line then $DEC$ is an isosceles triangle.

* Ex. **7**. If $AC = BC$, $\angle 1 = \angle 2$, and $CD$ and $CE$ are straight lines, then $\angle D = \angle E$.



Ex. 7

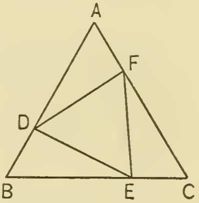
Ex. **8**. If the base of an isosceles triangle is trisected, the lines joining the points of division with the vertex are equal.

Ex. **9**. If in the sides of an equilateral $\triangle ABC$, the points $D$, $E$, and $F$ be taken so that $AD = BE = CF$, then $\triangle DFE$ is equilateral.

Ex. **10**. Lines drawn from the mid-points of the arms of an isosceles triangle to the mid-point of the base are equal.

Ex. **11**. If $DE = EF = FD$, and $\angle AFD = \angle BDE = \angle CEF$, then $\triangle ABC$ is equilateral.

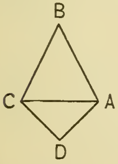
NOTE. The equality of angles is sometimes proved by means of Prop. IV (compare § 123).



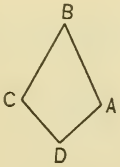
Exs. 9, 11

Ex. **12**. If $ABC$ and $ADC$ are two isosceles triangles on the same base, $AC$, then $\angle BAD = \angle BCD$.

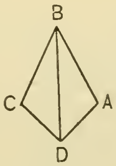
Ex. **13**. If in quadrilateral $ABCD$, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.



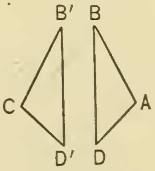
Ex. 12



Ex. 13



Ex. 14



Ex. 15

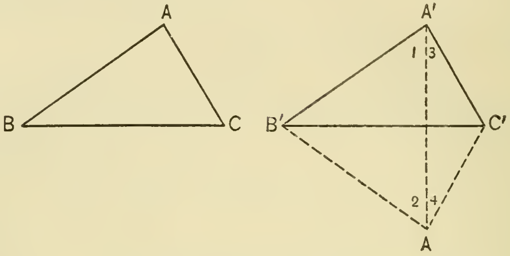
Ex. **14**. If two triangles $ABD$ and $DBC$ have $BD$ common, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.

Ex. **15**. If in two triangles $ABD$ and $CB'D'$, $AD = CD'$, $AB = CB'$, and $BD = B'D'$, then $\angle A = \angle C$.

HINT. Place the $\triangle$s together, so as to form a quadrilateral.

## PROPOSITION V. THEOREM

**74.** *Two triangles are congruent if three sides of one are equal respectively to three sides of the other* (*s. s. s. = s. s. s.*).



**Given** $\triangle$s $ABC$ and $A'B'C'$, with $AB = A'B'$, $BC = B'C'$, $AC = A'C'$.

**To prove** $\triangle ABC \cong \triangle A'B'C'$.

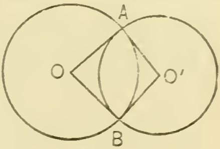
### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Place $\triangle ABC$ so that $BC$ shall coincide with $B'C'$ and $A$ and $A'$ lie on opposite sides of $B'C'$. | **1.** Axiom 19.<br>$BC = B'C'$, by hyp. |
| **2.** Draw $AA'$. | **2.** A st. line may be drawn between two points. |
| **3.** $\triangle AB'A'$ is isosceles. | **3.** $AB = A'B'$ by hyp. |
| **4.** $\therefore \angle 1 = \angle 2$. | **4.** The base $\angle$s of an isos. $\triangle$ are equal. |
| **5.** $\triangle AC'A'$ is isosceles. | **5.** $AC = A'C'$ by hyp. |
| **6.** $\therefore \angle 3 = \angle 4$. | **6.** Same as 4. |
| **7.** $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$. | **7.** If equals are added to equals, the sums are equal. |
| **8.** Or $\angle A = \angle A'$. | **8.** Sub. |
| **9.** $\therefore \triangle AB'C' \cong \triangle A'B'C$.<br>*i.e.* $\triangle ABC \cong \triangle A'B'C'$. | **9.** s. a. s. = s. a. s. |

Ex. 1. *The median to the base of an isosceles triangle bisects the vertex angle.*

Ex. 2. *If the opposite sides of a quadrilateral are equal, the opposite angles are equal.*

75. Method II. **If the lines or angles whose equality we wish to demonstrate are not parts of congruent triangles, we try to make them parts of congruent triangles by drawing additional lines.**



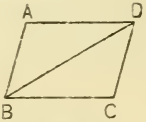
Exs. 1, 2

Ex. 1. If two circles whose centers are $O$ and $O'$ intersect in $A$ and $B$, then $\angle AOO' = \angle BOO'$.

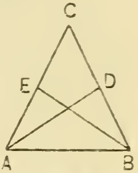
Ex. 2. If two circles whose centers are $O$ and $O'$ intersect in $A$ and $B$, then $\angle OAO' = \angle OBO'$.

Ex. 3. If $AB = CD$ and $BC = DA$, then $\angle ABD = \angle BDC$.

Ex. 4. If $AE = BD$, $AD = BE$, and $AC$ and $BC$ are st. lines, then $\angle CEB = \angle CDA$.



Ex. 3



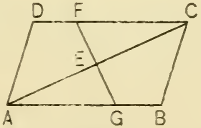
Ex. 4

76. Method III. **If it is impossible to prove the congruence of the required pair of triangles, prove first the congruence of some other pair, or pairs, whose corresponding parts will enable us to demonstrate the congruence of the original pair.**

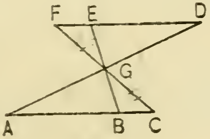
Ex. 5. If the opposite sides of a quadrilateral are equal, and a line be drawn through the mid-point of a diagonal, terminating in two sides, this line is bisected.

Ex. 6. If $AG = GD$, $CG = FG$, and all lines are straight lines, then $BG = GE$.

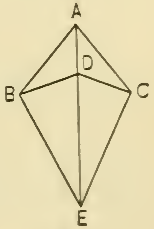
Ex. 7. If $AB = AC$, $AE$ bisects $\angle BAC$, and $AE$ is a st. line, then $\angle DBE = \angle DCE$.



Ex. 5



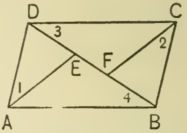
Ex. 6



Ex. 7

Ex. **8.** If $AB = CD$, $BC = DA$, and $\angle 1 = \angle 2$, and $DB$ is a st. line, then $AE = CF$.

Ex. **9.** If $AB = CD$, $\angle 3 = \angle 4$, and $BF = DE$, then $\angle 1 = \angle 2$.



Exs. 8, 9

Ex. **10.** Two triangles are congruent if two sides and the median to one of the sides are equal respectively to two sides and the corresponding median of the other.

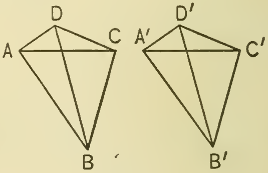
Ex. **11.** Two isosceles triangles are congruent if an arm and the line which joins the mid-points of the arms of one triangle are equal respectively to an arm and the line which joins the mid-points of the arms of the other.

Ex. **12.** *If the opposite sides of a quadrilateral are equal, the diagonals bisect each other.*

Ex. **13.** Two triangles $ABC$ and $A'B'C'$ are congruent if $AB = A'B'$, $\angle A = \angle A'$, and angle-bisector $AD$ = angle-bisector $A'D'$.

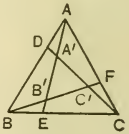
Ex. **14.** If in quadrilateral $ABCD$, $AB = BC$, $CD = DA$, and diagonals $AC$ and $BD$ meet in $E$, then $AE = EC$.

Ex. **15.** If in quadrilaterals $ABCD$ and $A'B'C'D'$, $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, and $AC = A'C'$, then $BD = B'D'$.



Ex. 15

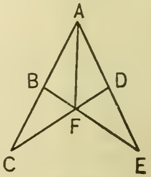
* Ex. **16.** If on the sides of an equilateral triangle $ABC$, the points $D$, $E$, $F$ are taken, so that $AD = BE = CF$, and $E$, $D$, and $F$ are joined to the opposite vertices, then $\triangle A'B'C'$ is equilateral.



Ex. 16

* Ex. **17.** If the opposite sides of a polygon of six sides are equal, and two of the opposite angles are equal, then all opposite angles are equal.

* Ex. **18.** If $AB = AD$, $AC = AE$, and $BE$ and $DC$ are st. lines, then $\angle BAF = \angle DAF$.



Ex. 18

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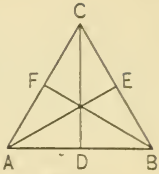
* See footnote on page 20.

* Ex. **19**. If $\angle A = \angle B$, and $AF = BE$, then $AD = DB$.



Ex. 19

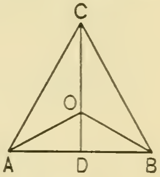
**77.** METHOD IV. **To prove that an angle is a right angle we usually demonstrate that it is equal to its supplementary adjacent angle.**

Ex. **20**. *The diagonals of an equilateral quadrilateral are perpendicular to each other.*

Ex. **21**. *The median to the base of an isosceles triangle is perpendicular to the base.*

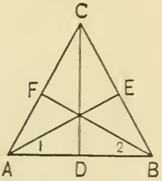
Ex. **22**. In quadrilateral $ABCD$, if $AB = BC$, and $CD = DA$, then $BD \perp AC$.

Ex. **23**. If $AC = BC$, and $AO = BO$, then $CD \perp AB$.



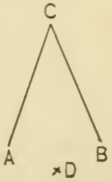
Ex. 23

* Ex. **24**. If $AC = BC$,
and $\angle 1 = \angle 2$,
then $CD \perp AB$.



Ex. 24

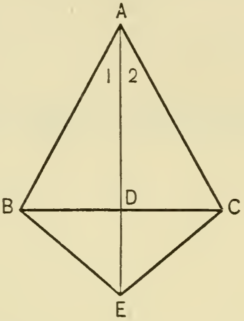
**78.** A point $C$ is said to be **equidistant** from two other points $A$ and $B$ if $AC = BC$.



It is not necessary that these two lines be drawn. Thus, we may say $D$ is equidistant from $A$ and $B$, even if $DA$ and $DB$ are not drawn. In general, two letters, as $AD$, mean the straight line connecting $A$ and $D$.

### PROPOSITION VI. THEOREM

**79.** *Two points each equidistant from the ends of a line, determine the perpendicular bisector of the line.*



**Given** line $BC$ and points $A$ and $E$ such that $AB = AC$, $BE = EC$.

**To prove** $AE$ is the perpendicular bisector of $BC$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** $AB = AC$, $BE = EC$. | **1.** Hyp. |
| **2.** $AE = AE$. | **2.** Iden. |
| **3.** $\triangle ABE \cong \triangle AEC$. | **3.** s. s. s. = s. s. s. |
| **4.** $\angle 1 = \angle 2$. | **4.** Corr. $\angle$s of $\cong$ $\triangle$s are =. |
| **5.** $\triangle ABC$ is isosceles. | **5.** $AB = AC$, by hyp. |
| **6.** St. line $ADE$ is the $\perp$ bisector of $BC$. | **6.** The bisector of the vertex $\angle$ of an isos. $\triangle$ is $\perp$ to the base and bisects the base. |

**80.** COR. *Every point equidistant from the ends of a line lies in the perpendicular bisector of that line.*

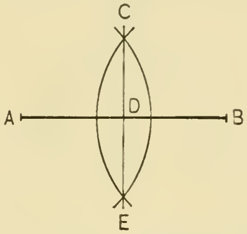
---

EX. *If the four sides of a quadrilateral are equal, the diagonals bisect each other.*

## CONSTRUCTIONS

### Proposition VII. Problem

**81.** *To bisect a given straight line.*



**Given** a straight line $AB$.

**Required** to bisect $AB$.

**Construction.** From $A$ and $B$ as centers, with equal radii greater than $\frac{1}{2} AB$, describe arcs intersecting at $C$ and $E$.

Draw $CE$.

Then the line $CE$ bisects $AB$ at $D$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** $C$ is equidistant from $A$ and $B$. | **1.** Con. |
| **2.** $E$ is equidistant from $A$ and $B$. | **2.** Con. |
| **3.** $\therefore AD = DB$. | **3.** Two points each equidistant from the ends of a line determine the perpendicular-bisector to that line. |

**82.** Cor. By means of the preceding construction we can construct the perpendicular bisector ($CE$) of a given line ($AB$).

---

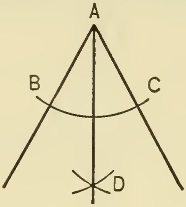
Ex. **1.** *Divide a given line into four equal parts.*

Ex. **2.** *Construct the three medians of a triangle.*

Ex. **3.** *Construct the three perpendicular bisectors of the three sides of a triangle.*

### PROPOSITION VIII. PROBLEM

**83.** *To bisect a given angle.*



**Given** $\angle CAB$.

**Required** to bisect $\angle CAB$.

**Construction.** With $A$ as a center and any radius, as $AB$, describe an arc cutting the sides of the $\angle A$ at $B$ and $C$.

From $B$ and $C$ as centers, with equal radii greater than one half the distance from $B$ to $C$, describe two arcs intersecting at $D$. Draw $AD$.

$AD$ is the required bisector.

HINT. Prove $AD$ is the bisector by first drawing $BD$ and $DC$ and then proving two ≜ are congruent.

**84.** It is advisable to draw the lines of construction either very thin or dotted, while the given and required lines are represented by heavy full lines. In complex constructions the resulting lines may be drawn in red or in blue.

---

Ex. **1.** *To divide a given angle into four equal parts.*

Ex. **2.** *To bisect a straight angle.*

Ex. **3.** *Construct an angle of 90°, of 45°.*

Ex. **4.** Construct an angle of 22° 30′.

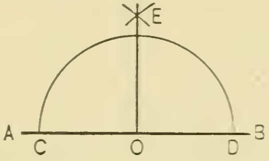
Ex. **5.** Construct an angle of 67° 30′.

Ex. **6.** Construct one half the supplement of a given angle.

Ex. **7.** *Construct the angle-bisectors of a given triangle.*

## PROPOSITION IX. PROBLEM

**85.** *At a given point in a given straight line, to erect a perpendicular to that line.*



**Given** point $O$ in line $AB$.

**Required** the perpendicular to the line $AB$ at $O$.

**Construction.** From $O$ as a center with any radius $OC$, describe an arc intersecting $AB$ in $C$ and $D$.

From $C$ and $D$ as centers with equal radii greater than $OC$, describe two arcs intersecting at $E$.

Draw $OE$.

$OE$ is the required perpendicular.

NOTE 1. Draw $CE$ and $DE$. Prove two $\triangle$ are congruent. Then show that $OE$ is perpendicular to $AB$ at $O$. See prop. 7, p. 37 for model for proof.

NOTE 2. Whenever a pupil has carried out a construction, he should prove the correctness of his construction.

---

Ex. **1.** *Construct the complement of a given acute angle.*

Ex. **2.** Construct the supplement of the complement of a given acute angle.

Ex. **3.** Construct one half the complement of a given acute angle.

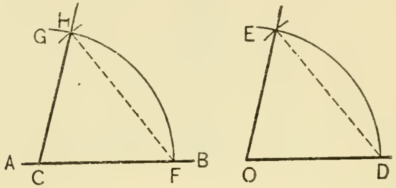
Ex. **4.** Construct a right triangle given the arms equal to 1″ and $1\frac{1}{2}$″ respectively.

Ex. **5.** Construct a square having a side equal to $1\frac{1}{3}$″.

Ex. **6.** Construct a square given its perimeter equal to a given line.

### PROPOSITION X. PROBLEM

**86.** *Through a given point on a straight line to draw a line making a given angle with the given line.*



**Given** point $C$ in line $AB$ and $\angle O$.

**Required** an $\angle$ at $C = \angle O$.

**Construction.** From $O$ as a center with any radius, describe an arc cutting the sides of $\angle O$ in $D$ and $E$.

From $C$ as a center, with the same radius, describe arc $FG$, intersecting $CB$ in $F$. From $F$ as a center, with a radius equal to $DE$, draw an arc intersecting arc $FG$ in $H$. Draw $CH$.

$\angle HCF$ is the required angle.

HINT. Prove this proposition by showing the two △ are congruent.

NOTE. Two letters, *e.g.* $DE$, used as above, denote a straight line.

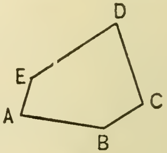
---

EX. **1.** *Construct an angle equal to twice a given angle.*

EX. **2.** Construct an angle equal to twice the supplement of a given angle.

EX. **3.** Given a triangle $ABC$. Construct another triangle $A'B'C'$, having $A'B' = AB$, $\angle A' = \angle A$, and $\angle B' = \angle B$. What is the relation of △ $ABC$ and $A'B'C'$?

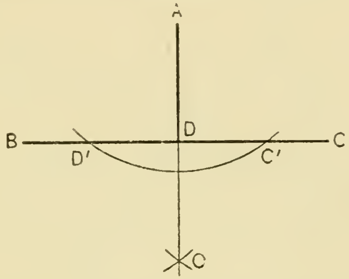
EX. **4.** Given a polygon of five sides (pentagon) $ABCDE$. Construct a pentagon having four of its sides equal respectively to $AB$, $BC$, $CD$, and $DE$, and the included angles respectively equal to ∡$B$, $C$, and $D$.



EX. 4

### PROPOSITION XI. PROBLEM

**87.** *From a point without a straight line, to let fall a perpendicular upon that line.*



**Given** a straight line $BC$, and a point $A$ without the line.

**Required** a perpendicular from the point $A$ to the line $BC$.

**Construction.** From $A$ as a center, with a radius sufficiently great, describe an arc cutting $BC$ in $C'$ and $D'$.

From $D'$ and $C'$ as centers, with equal radii greater than $\frac{1}{2}$ $D'C'$, describe two arcs intersecting at $O$.

Draw $AO$ intersecting $BC$ in $D$.

$AD$ is the required perpendicular.

[The proof is left to the student. Use § 79.]

NOTE. We can infer from this proposition that *from a point without a line one perpendicular can be drawn to the given line.*

---

Ex. **1.** *Construct the three altitudes of an acute triangle.*

Ex. **2.** *Construct the three altitudes of an obtuse triangle; of a right triangle.*

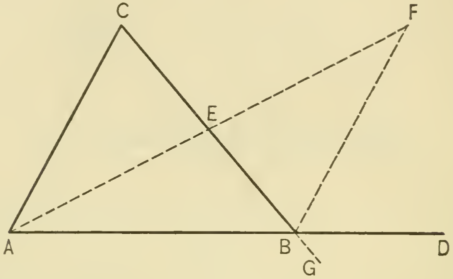
Ex. **3.** From your construction of the three altitudes of a triangle, what can you say about these three lines?

Ex. **4.** About what other lines in a triangle can you say the same thing?

Ex. **5.** Construct a right triangle given hypotenuse and an angle of 45°.

## PROPOSITION XII. THEOREM

**88.** *An exterior angle of a triangle is greater than either remote interior angle.*



**Given** $\triangle ABC$ and the ext. $\angle CBD$.

**To prove** $\angle CBD > \angle C$ or $\angle A$.

### Proof

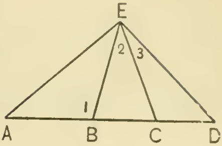
| STATEMENTS | REASONS |
|---|---|
| 1. Let $E$ be the mid-point of $BC$. Draw $AE$ and produce it its own length to $F$. Draw $FB$. | 1. Every line has a bisector.<br>A st. line can be drawn between two points.<br>A line may be extended. |
| 2. In $\triangle$s $ACE$ and $FBE$, $AE = EF$. | 2. Con. |
| 3. $EC = BE$. | 3. Con. |
| 4. $\angle CEA = \angle FEB$. | 4. Vertical $\angle$s are equal. |
| 5. $\triangle ACE \cong \triangle FBE$. | 5. s. a. s. = s. a. s. |
| 6. $\angle EBF = \angle C$. | 6. Corr. parts of $\cong$ $\triangle$s are =. |
| 7. $\angle CBD > \angle EBF$. | 7. The whole is greater than any of its parts. |
| 8. Hence $\angle CBD > \angle C$. | 8. Sub. |

NOTE. By joining the mid-point of $AB$ to $C$, etc., it follows that

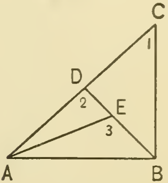
$$\angle ABG > \angle A,\quad \angle ABG = \angle CBD,$$

$$\therefore \angle CBD > \angle A.$$

Ex. **1.** If $AD$ is a st. line, prove that (*a*) $\angle 1 > \angle 2$, (*b*) $\angle 1 > \angle D$, (*c*) $\angle 1 > \angle 3$.



Ex. 1



Ex. 2

Ex. **2.** If $DB$ is a st. line, prove that (*a*) $\angle 2 > \angle 1$, (*b*) $\angle 3 > \angle 1$.

Ex. **3.** In the diagram for Proposition XII, prove: (*a*) $\angle FBD > \angle F$, (*b*) $\angle BEA > \angle ACE$, (*c*) $\angle CEA > \angle GBD$, (*d*) $\angle CBD > \angle CEF$.

**89.** Def. A **transversal** is a line that intersects two (or more) other lines. The various angles formed by two lines and a transversal are named as follows:

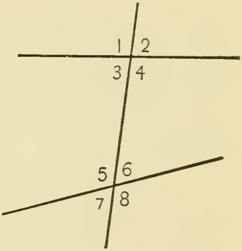
1, 2, 7, 8 are **exterior** angles.

3, 4, 5, 6 are **interior** angles.

1 and 8, 2 and 7 are **alternate exterior** angles.

3 and 6, 4 and 5 are **alternate interior** angles.

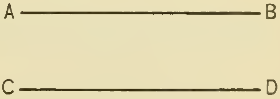
1 and 5, 2 and 6, 3 and 7, 4 and 8 are **corresponding** angles.



## PARALLEL LINES

**90.** Def. Two lines are **parallel** if they lie in the same plane and do not meet, however far produced (in each direction).

Thus, $AB$ and $CD$ represent two parallel lines.



**91.** Axiom 16. *Two intersecting lines cannot both be parallel to a third straight line.*

Note. We assume that *two unlimited straight lines in the same plane either intersect or are parallel.*

**92.** Theorem. *Two straight lines which are parallel to a third straight line are parallel to each other.*

For if the two lines should meet, we should have two intersecting straight lines parallel to a third straight line, which contradicts Axiom 16.

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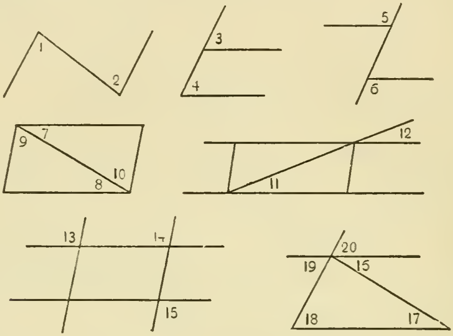
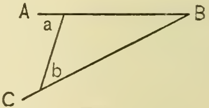


Ex. **1.** In the diagrams given above, tell what kind of angles the following are: 1 and 2, 3 and 4, 5 and 6, 7 and 8, 9 and 10, 11 and 12, 13 and 14, 14 and 15, 16 and 17, 18 and 19, 18 and 20.

Ex. **2.** If $AB$ and $BC$ are st. lines, is it possible that

(*a*) $\angle a = 60°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 70°$?
(*c*) $\angle a = 60°$, and $\angle b = 60°$?



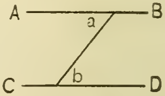
Ex. 2

Ex. **3.** Is it possible that the prolongations of $AB$ and $CD$ (*i.e.* toward the right) meet if

(*a*) $\angle a = 60°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 70°$?
(*c*) $\angle a = 60°$, and $\angle b = 60°$?

Ex. **4.** Do the prolongations of $BA$ and $DC$ (*i.e.* toward the left) meet if

(*a*) $\angle a = 60°$, $\angle b = 70°$?
(*b*) $\angle a = 60°$, $\angle b = 50°$?
(*c*) $\angle a = 60°$, $\angle b = 60°$?



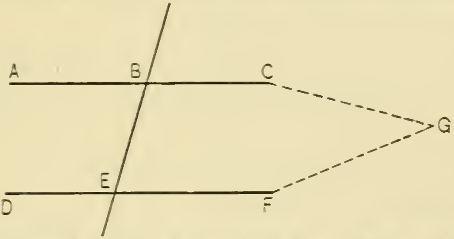
Exs. 3–5

Ex. **5.** Can the lines produced meet at all if

(*a*) $\angle a = 50°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 60°$?

## PROPOSITION XIII. THEOREM

**93.** *Two lines are parallel if a transversal to these lines makes a pair of alternate interior angles equal.*[1]



Given $AC$ and $DF$, intersected by $BE$ so that $\angle ABE = \angle BEF$.

To prove $AC \parallel DF$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. $AC \parallel DF$ or $AC$ not $\parallel DF$. | 1. Only two suppositions are admissible. Sect. 91. Note. |
| 2. If $AC$ and $DF$ are not parallel, they meet at a pt. $G$, forming $\triangle EBG$. | 2. Def. of nonparallel lines. |
| 3. $\angle ABE > \angle BEG$. | 3. Ext. $\angle$ of a $\triangle >$ either remote interior angle. |
| 4. This is impossible. | 4. By hyp., $\angle ABE = \angle BEG$. |
| 5. $\therefore AC \parallel DF$. | 5. The other supposition has been proved false. |

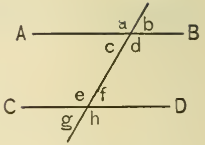
**94.** METHOD V. To demonstrate that two lines are parallel, prove the equality of a pair of alternate interior angles.

---

[1] In this proof we make a wrong assumption, viz., that two lines meet if the alt. int. ∠s are equal. The diagram which illustrates this wrong assumption must necessarily be inaccurate, and hence the two angles are not exactly equal. Similarly, all proofs which require the investigation of a wrong assumption (the so-called indirect proofs) are illustrated by inaccurate drawings, and it is illogical to attempt to make these drawings exact.

Ex. **1.** Prove that $AB$ and $CD$ are parallel if

(*a*) $\angle c = 70^\circ$, $\angle f = 70^\circ$.
(*b*) $\angle c = 60^\circ$, $\angle e = 120^\circ$.
(*c*) $\angle a = 110^\circ$, $\angle f = 70^\circ$.
(*d*) $\angle b = 60^\circ$, $\angle f = 60^\circ$.
(*e*) $\angle a = 120^\circ$, $\angle g = 60^\circ$.



Exs. 1–4

Ex. **2.** Prove that $AB \parallel CD$, if $\angle b = \angle f$.

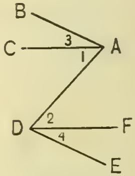
Ex. **3.** Prove that $AB \parallel CD$, if $\angle a = \angle h$.

Ex. **4.** Prove that $AB \parallel CD$, if $\angle b = \angle g$.

Ex. **5.** If $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, prove that $AB \parallel DE$.

Ex. **6.** If $\angle A = \angle D$, and $\angle 3 = \angle 4$, prove that $AC \parallel DF$.

Ex. **7.** If $BA \perp AD$, $ED \perp AD$, and $\angle 3 = \angle 4$, prove that $AC \parallel DF$.

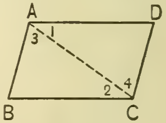


Exs. 5–7

Ex. **8.** If $AB$ and $CD$ bisect each other in $E$, prove that $AC \parallel DB$.

Ex. **9.** If $AB = DC$, and $\angle 3 = \angle 4$, prove that $AD \parallel BC$.

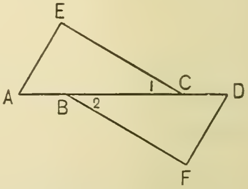
Ex. **10.** If $AB = DC$, and $AD = BC$, prove that $AB \parallel DC$.



Exs. 9, 10

Ex. **11.** If $AB = CD$, $EC = BF$, $\angle 1 = \angle 2$, and $AD$ is a st. line, then $AE \parallel DF$.

Ex. **12.** If $AE = DF$, $AB = CD$, and $EC = BF$, then $EC \parallel BF$.

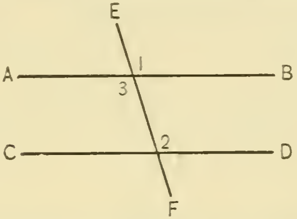


Exs. 11–13

Ex. **13.** If $AB = CD$, $\angle ABF = \angle DCE$, $EC = BF$, and $AD$ is a straight line, then $AE \parallel DF$.

### PROPOSITION XIV. THEOREM

**95.** *Two lines are parallel if a transversal to these lines makes a pair of corresponding angles equal.*



Given $AB$ and $CD$, intersected by $EF$ so that $\angle 1 = \angle 2$.

To prove $AB \parallel CD$.

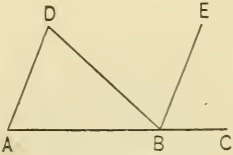
### Proof

| | STATEMENTS | REASONS |
|---|---|---|
| 1. | $\angle 1 = \angle 2$. | 1. Hyp. |
| 2. | $\angle 1 = \angle 3$. | 2. Vert. ∠s are =. |
| 3. | $\angle 2 = \angle 3$. | 3. Things = to the same thing are = to each other. |
| 4. | $\therefore AB \parallel CD$. | 4. Two lines are ∥ if a pair of alt. int. ∠s are =. |

**96.** COR. *If two lines are perpendicular to a third line, they are parallel.*

---

Ex. 1. In the diagram for Ex. 1, p. 46, if $\angle a = 2\angle b$, and $\angle f = 60°$, then $AB \parallel CD$.



Exs. 2–4

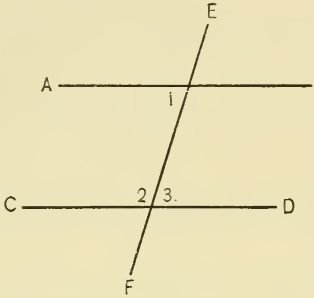
Ex. 2. If $\angle A = 70°$, $\angle ABD = 60°$, $\angle DBE = 50°$, and $AC$ is a straight line, prove that $AD \parallel BE$.

Ex. 3. If $BE$ bisects $\angle DBC$, $\angle ABD = 30°$, $\angle A = 75°$, and $AC$ is a straight line, then $AD \parallel BE$.

Ex. 4. If $AB = BD$ and $\angle D = \angle EBC$, then $AD \parallel BE$.

## Proposition XV. Theorem

**97.** *Two lines are parallel if a transversal to these lines makes a pair of interior angles on the same side of the transversal supplementary.*



**Given** *AB* and *CD*, intersected by *EF* so that ∠1 and ∠2 are supplementary.

**To prove** *AB* ∥ *CD*.

**Proof**

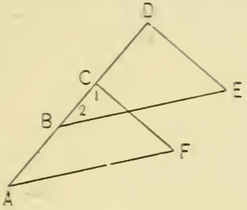
| STATEMENTS | REASONS |
|---|---|
| **1.** ∠1 is the sup. of ∠2. | **1.** Hyp. |
| **2.** ∠3 is the sup. of ∠2. | **2.** If two adj. ∠s have their ext. sides in a st. line, they are sup. |
| **3.** ∠1 = ∠3. | **3.** Supplements of the same ∠ are equal. |
| **4.** ∴ *AB* ∥ *CD*. | **4.** Lines are ∥ if a pair of alt. int. ∠s are equal. |

**98.** Method VI. **Sometimes lines are demonstrated to be parallel by proving that:**

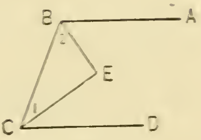
**(*a*) Two corresponding angles are equal, or**

**(*b*) Two interior angles on the same side of a transversal are supplementary.**

Ex. 1. If $AF = BE$, $AB = CD$, $\angle A = \angle 2$, and $AD$ is a st. line, prove that $CF \parallel DE$.



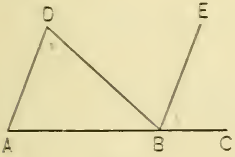
Ex. 1



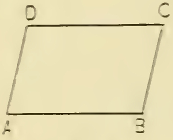
Ex. 2

Ex. 2. If $BE$ bisects $\angle B$, $CE$ bisects $\angle C$, and $\angle 1 + \angle 2 = 90°$, then $BA \parallel CD$.

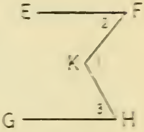
Ex. 3. If $AB = BD$ and $\angle D$ is the supplement of $\angle ABE$, then $AD \parallel BE$.



Ex. 3



Ex. 4



Ex. 5

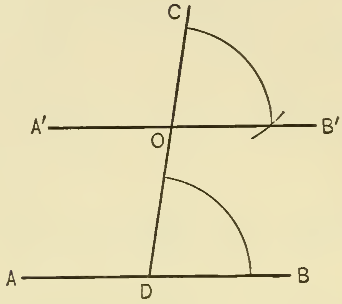
Ex. 4. If $\angle A + \angle B + \angle C + \angle D = 360°$, $\angle A = \angle C$, and $\angle B = \angle D$, then $AD \parallel BC$.

* Ex. 5. If $\angle 1 = \angle 2 + \angle 3$, then $EF \parallel GH$

**99.** To **construct figures** whose parts bear certain relations to each other, we employ the same methods as for proving these relations. Thus, to draw a line parallel to another, we may construct a pair of equal alt. int. ∠s; to make an ∠ equal to another ∠ we use a construction (86) which really involves the construction of congruent △s, etc.

### PROPOSITION XVI. PROBLEM

**100.** *Through a given point to draw a parallel to a given line.*



**Given** line $AB$ and point $O$.

**Required** the line through $O \parallel AB$.

**Construction.** Through $O$ draw $DC$, intersecting $AB$ in $D$.
At $O$ construct $\angle COB' = \angle ODB$. (86)
$A'B$ is the required line.

HINT. What are the means of proving that two lines are parallel?

NOTE. If previous constructions are employed in a problem, the details of such construction should *not* be stated. Thus, in the preceding problem it would be awkward and illogical to describe the method of making $\angle COB' = \angle ODB$.

The beginner, however, should *draw* all such details. In other words the construction should be carried out exactly by means of ruler and compasses.

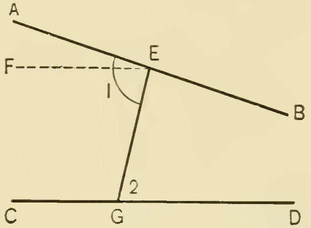
---

Ex. 1. Through a given point to draw a line parallel to a given line, by means of equal alt. int. ∠s.

Ex. 2. Through a given point to draw a line parallel to a given line by means of Ex. 8, p. 46.

## PROPOSITION XVII. THEOREM

**101.** *Two lines are not parallel if a transversal to these lines makes a pair of alternate interior angles unequal.*



**Given** *AB* and *CD* cut by a transversal, so that

$$\angle 1 > \angle 2.$$

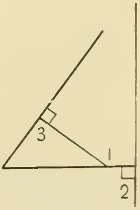
**To prove** *AB* is not $\parallel$ *CD*.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw *EF* so that $\angle FEG = \angle 2$. | **1.** To draw a line to make a given $\angle$ with a given line. |
| **2.** $EF \parallel CD$. | **2.** Two lines are $\parallel$ if a transversal makes the alt. int. $\angle$s equal. |
| **3.** $\therefore$ *AB* is not $\parallel$ *CD*. | **3.** Two intersecting lines cannot both be $\parallel$ to a third line. |

**102.** COR. *Two lines respectively perpendicular to two intersecting lines are not parallel.*

HINT. Prove $\angle 1 > \angle 2$.

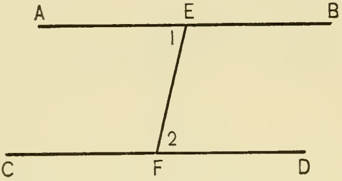


**103.** A proposition is the **converse** of another when the hypothesis and the conclusion of the one are respectively the conclusion and the hypothesis of the other.

### PROPOSITION XVIII. THEOREM

**104.** *If two parallels are cut by a transversal, the alternate interior angles are equal.*

[Converse of Prop. XIII.]



**Given** parallel lines $AB$ and $CD$, and the alt. int. ∡ 1 and 2.

**To prove** $\angle 1 = \angle 2$.

**Proof**

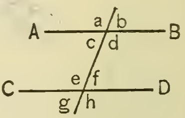
| STATEMENTS | REASONS |
|---|---|
| **1.** Either $\angle 1 = \angle 2$, or $\angle 1 \neq \angle 2$. | **1.** Only two suppositions are admissible. |
| **2.** Suppose that $\angle 1 \neq \angle 2$, then $AB$ is not $\parallel$ to $CD$. | **2.** Two lines are not $\parallel$ if a transversal to these lines makes a pair of alt. int. ∡ unequal. |
| **3.** But this is impossible. | **3.** $AB \parallel CD$, by hyp. |
| **4.** $\therefore \angle 1 = \angle 2$. | **4.** The other supposition has been proved false. |

**105.** COR. *If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

---

Ex. **1.** Two $\parallel$ lines $AB$ and $CD$ are cut by a transversal.

(*a*) If $\angle c = 70°$, find $\angle f$.
(*b*) If $\angle c = 80°$, find $\angle e$.
(*c*) If $\angle c = 75°$, find $\angle g$.



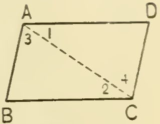
Exs. 1–3

Ex. 2. If $AB \parallel CD$, and
(a) $\angle a = 110°$, find $\angle f$.
(b) $\angle b = 80°$, find $\angle g$.
(c) $\angle h = 100°$, find $\angle b$.

Ex. 3. If $AB \parallel CD$, prove that
(a) $\angle a = \angle e$.
(b) $\angle b = \angle f$.
(c) $\angle a = \angle h$.
(d) $\angle c$ is the sup. of $\angle e$.
(e) $\angle g$ is the sup. of $\angle d$.

Ex. 4. If $AD = BC$, and $AD \parallel BC$, then $\angle 3 = \angle 4$.

Ex. 5. If $AB \parallel CD$, and $AD \parallel BC$, then $\angle B = \angle D$.

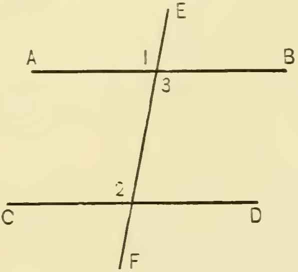


Exs. 4, 5

### PROPOSITION XIX. THEOREM

**106.** *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

[Converse of Prop. XIV.]



**Given** parallel lines $AB$ and $CD$ and the corr. ∠s 1 and 2.

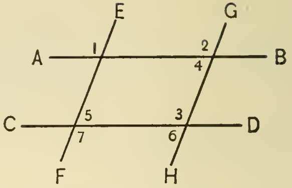
**To prove** $\angle 1 = \angle 2$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** $\angle 1 = \angle 3$. | 1. Vertical ∠s are equal. |
| **2.** $\angle 2 = \angle 3$. | 2. Alt. int. ∠s of $\parallel$ lines are equal. |
| **3.** $\therefore \angle 1 = \angle 2$. | 3. Things equal to the same thing are equal to each other. |

Ex. **1**. If $AB \parallel CD$ and $EF \parallel GH$, prove that

(*a*) $\angle 1 = \angle 3$.
(*b*) $\angle 4 = \angle 5$.
(*c*) $\angle 2 = \angle 7$.



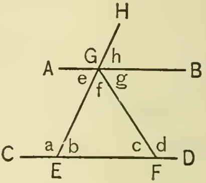
Ex. 1

Ex. **2**. If $AB \parallel CD$, and $EH$ is a st. line, find the numerical value of

(*a*) $\angle d$, if $\angle g = 80°$.
(*b*) $\angle a$, if $\angle h = 60°$.
(*c*) $\angle FGH$, if $\angle b = 60°$, and $\angle c = 70°$.
(*d*) $\angle b$, if $\angle FGH = 140°$, and $\angle c = 80°$.
(*e*) $\angle f$, if $\angle b = 65°$, and $\angle c = 70°$.
(*f*) $\angle b$, if $\angle f = 50°$, and $\angle c = 70°$.
(*g*) $\angle HGF$, if $\angle e = 55°$, and $\angle c = 75°$.

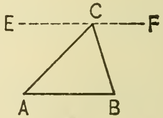
Ex. **3**. If $AB \parallel CD$, prove that

(*a*) $\angle FGH = \angle b + \angle c$.
(*b*) $\angle f = 180° - \angle h - \angle c$.
(*c*) $\angle CEH = \angle f + \angle c$.
(*d*) $\angle b + \angle f + \angle c = 180°$.



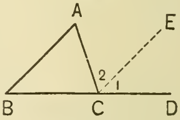
Exs. 2, 3

Ex. **4**. Prove that the sum of the three angles of $\triangle ABC$ equals 180°.



Ex. 4

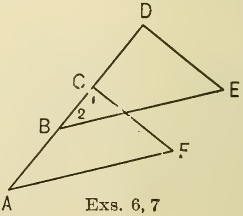
Ex. **5**. If $ACD$ is an ext. angle of $\triangle ABC$, then $\angle ACD = \angle A + \angle B$.



Ex. 5

Ex. **6**. If $AD$ is a st. line, $AB = CD$, $AF \parallel BE$, and $CF \parallel DE$, then $DE = CF$.

Ex. **7**. If $AB = CD$, $CF = DE$, $CF \parallel DE$, and $AD$ is a st. line, then $AF = BE$.

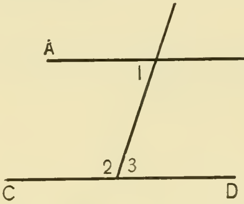


Exs. 6, 7

## PROPOSITION XX. THEOREM

**107.** *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

[Converse of Prop. XV.]



**Given** $AB \parallel CD$, and the int. $\angle$s 1 and 2, lying on the same side of a transversal.

**To prove** $\angle 1$ is the sup. of $\angle 2$.

### Proof

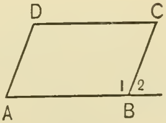
| STATEMENTS | REASONS |
|---|---|
| **1.** $AB \parallel CD$. | **1.** By hyp. |
| **2.** $\angle 1 = \angle 3$. | **2.** Alt. int. $\angle$s of $\parallel$s are $=$. |
| **3.** $\angle 3$ is sup. of $\angle 2$. | **3.** If two adj. $\angle$s have their exterior sides in a st. line, they are sup. |
| **4.** $\therefore \angle 1$ is sup. of $\angle 2$. | **4.** Sub. |

---

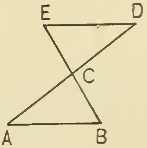
Ex. **1.** If $AB \parallel CD$, and $AD \parallel BC$, prove

(*a*) $\angle 2$ is the sup. of $\angle D$.
(*b*) $\angle A = \angle C$.

Ex. **2.** If $ED \parallel AB$, $AC = CD$, and $EB$ and $AD$ are st. lines, $\triangle ABC \cong \triangle CDE$.



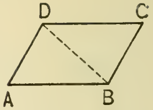
Ex. 1



Ex. 2

Ex. **3.** If the opposite sides of a quadrilateral are parallel, they are also equal.

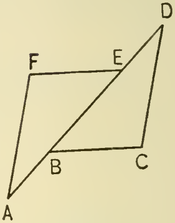
Ex. **4.** If two sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel.



Exs. 3, 4

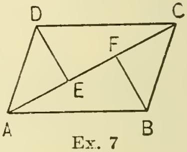
Ex. **5.** If $AB = ED$, $BC \parallel FE$, $AF \parallel DC$, and $AD$ is a st. line, $\triangle AEF$ must be congruent to $\triangle BCD$.

Ex. **6.** If $AB = ED$, $BC = FE$, $BC \parallel FE$, and $AD$ is a st. line, $AF$ must equal $CD$.



Exs. 5, 6

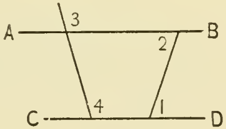
Ex. **7.** If $AB \parallel CD$, $AD \parallel BC$, $AE = FC$, and $AC$ in a st. line, then $DE \parallel BF$. (Apply Ex. 3.)



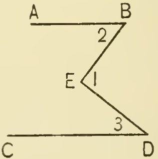
Ex. 7

Ex. **8.** The bisectors of a pair of corresponding angles formed by parallel lines are parallel.

Ex. **9.** In the diagram below, if $\angle 1 = \angle 2$, then $\angle 3 = \angle 4$.



Ex. 9

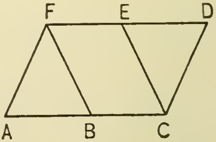


Ex. 10

* Ex. **10.** If in the diagram above, $AB \parallel CD$, prove that $\angle 1 = \angle 2 + \angle 3$.

* Ex. **11.** If in the diagram, $AC \parallel FD$, $AF \parallel CD$, and $FB \parallel EC$, prove that

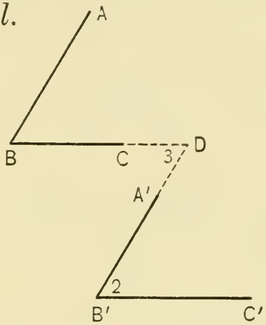
$$\triangle AFB \cong \triangle ECD.$$



Ex. 11

## PROPOSITION XXI. THEOREM

**108.** *Two angles whose corresponding sides are parallel and extend in the same direction from their vertices are equal.*



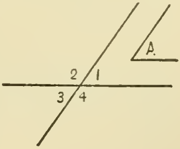
**Given** $\angle B$ and $\angle B'$ with $AB \parallel A'B'$ and $BC \parallel B'C'$.
**To prove** $\angle B = \angle B'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. $AB \parallel A'B'$ and $BC \parallel B'C'$. | 1. Hyp. |
| 2. Produce, if necessary, $BC$ and $B'A'$ until they meet at $D$. | 2. Two intersecting lines cannot both be $\parallel$ to a third st. line. |
| 3. $\angle B = \angle 3$ and $\angle B' = \angle 3$. | 3. Alt. int. $\measuredangle$ of $\parallel$ lines are =. |
| 4. $\therefore \angle B = \angle B'$. | 4. Things = to the same thing are = to each other. |

**109.** COR. *Angles whose corresponding sides are parallel are either equal or supplementary.*

Thus, in the diagram opposite, if the corresponding sides are parallel, $\measuredangle$ 1 and 3 equal $\angle A$, and $\measuredangle$ 2 and 4 are sup. of $\angle A$.

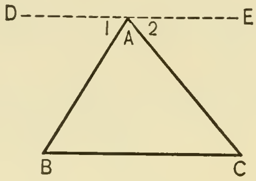


NOTE. Two $\measuredangle$ are equal if their sides are parallel right side to right side and left side to left side.

## TRIANGLES — PART II

### PROPOSITION XXII. THEOREM

**110.** *The sum of the angles of a triangle is equal to a straight angle.*



**Given** $\triangle ABC$.

**To prove** $\angle A + \angle B + \angle C =$ a st. $\angle$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Through $A$ draw $DE \parallel BC$. | **1.** Through a given pt. to draw a line $\parallel$ to a given line. |
| **2.** $\angle 1 = \angle B$. | **2.** Alt. int. ∡ of $\parallel$ lines are =. |
| **3.** $\angle 2 = \angle C$. | **3.** Alt. int. ∡ of $\parallel$ lines are =. |
| **4.** $\angle A + \angle 1 + \angle 2 =$ a st. $\angle$. | **4.** The whole equals the sum of its parts. |
| **5.** $\therefore \angle A + \angle B + \angle C =$ a st. $\angle$. | **5.** Sub. |

**111.** COR. 1. *In a triangle there can be at most one obtuse angle or one right angle.*

**112.** COR. 2. *The acute angles of a right triangle are complementary.*

**113.** COR. 3. *If two triangles have two angles of the one equal respectively to two angles of the other, the third angles are equal.*

**114.** Cor. 4. *Two triangles are congruent if two angles and the side opposite one of them are equal respectively to two angles and the corresponding side of the other* (s. a. a. = s. a. a.).

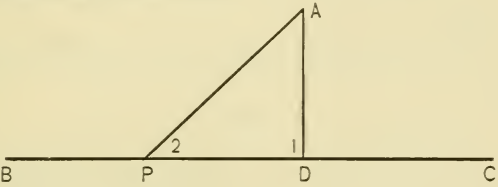
**115.** Cor. 5. *Each angle of an equiangular triangle is 60°.*

**116.** Cor. 6. *Two right triangles are congruent, if the hypotenuse and an acute angle of one triangle are equal to the hypotenuse and an acute angle of the other triangle.*

Hint. Use Cor. 4.

**117.** Cor. 7. *Two right triangles are congruent, if an arm and an acute angle of one triangle are equal to an arm and the corresponding acute angle of the other triangle.*

**118.** Cor. 8. *From a point without a line there can be only one perpendicular to that line.*



**Given** point $A$ outside line $BC$, $AD \perp BC$, $AP$ any other line from $A$ to $BC$.

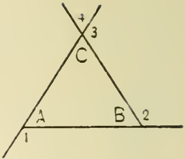
**To prove** $AP$ is not $\perp$ to $BC$.

## Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** At least one $\perp$ $AD$ can be drawn to $BC$. | **1.** From a pt. without a line, to let fall a $\perp$ to that line (Prop. XI). |
| **2.** $\angle 1$ is a rt. $\angle$. | **2.** Def. of $\perp$. |
| **3.** $\angle 2$ is an acute $\angle$. | **3.** A right $\triangle$ has two acute ∡. |
| **4.** $\therefore$ $AP$ is not $\perp$ $BC$. | **4.** $AP$ does not form rt. ∡ with $BC$. |

**Ex. 1.** Find the value of

(*a*) $\angle A$, if $\angle B = 60°$, and $\angle C = 50°$.
(*b*) $\angle B$, if $\angle A = m°$, and $\angle C = n°$.
(*c*) $\angle 3$, if $\angle A = 50°$, and $\angle B = 70°$.
(*d*) $\angle 4$, if $\angle 2 = 110°$, and $\angle A = 60°$.
(*e*) $\angle 1$, if $\angle 4 = 40°$, and $\angle B = 70°$.
(*f*) $\angle 3$, if $\angle 2 = 140°$, and $\angle 1 = 120°$.
(*g*) $\angle A + \angle B$, if $\angle C = m°$.



Ex. 1

**Ex. 2.** If $C$ is the vertex angle of isosceles triangle $ABC$, find

(*a*) $\angle A$, if $\angle C = 40°$.
(*b*) $\angle B$, if $\angle C = m°$.
(*c*) $\angle C$, if $\angle A = 40°$.
(*d*) $\angle C$, if $\angle B = n°$.

**Ex. 3.** *How many degrees are there in each angle of an isosceles right triangle?*

**Ex. 4.** If two angles of a triangle are 60° and 40° respectively, what is the angle formed by the bisectors of these angles?

**Ex. 5.** *If two angles of a triangle are given, construct the third.*

**Ex. 6.** If a base angle of an isosceles triangle is given, construct the vertex angle.

**Ex. 7.** If the vertex angle of an isosceles triangle is given, construct a base angle.

**Ex. 8.** *Construct angles of* 60°; 30°; 120°; 75°.

**Ex. 9.** The bisectors of two interior angles on the same side of a transversal to two parallel lines are perpendicular to each other.

**Ex. 10.** If $CB$ and $DE$ are drawn respectively perpendicular upon the sides of angle $A$, prove $\angle ACB = \angle ADE$.

**Ex. 11.** If $CD$ is the altitude upon the hypotenuse of right triangle $ABC$, prove $\angle ACD = \angle B$.

**Ex. 12.** If $CD$ and $AE$ are altitudes of triangle $ABC$, prove $\angle BCD = \angle BAE$.

**Ex. 13.** *Find the sum of the four angles of a quadrilateral.*

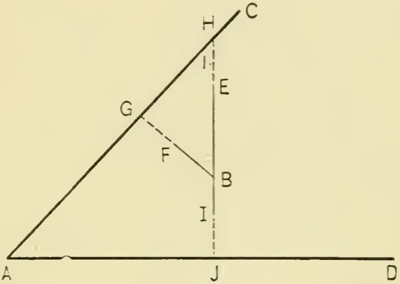
**Ex. 14.** If two angles of a triangle are equal, the bisector of the third angle divides the figure into two congruent triangles.

**Ex. 15.** The altitudes upon the arms of an isosceles triangle are equal.

**Ex. 16.** The lines drawn from the mid-point of the base of an isosceles triangle perpendicular to the arms are equal.

### PROPOSITION XXIII. THEOREM

**119.** *If two angles have their sides respectively perpendicular, they are either equal or supplementary.*



**Given** ∡ $CAD$ and $FBE$ with $AD \perp BE$, $AC \perp BF$.

**To prove** $\angle CAD = \angle FBE$ and $\angle CAD$ suppl. to $\angle FBI$.

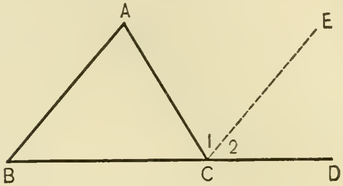
#### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Extend $BF$ to $G$, $BE$ to $H$, and $BI$ to $J$. | **1.** A line may be extended. |
| **2.** ∡ $BGH$ and $AJH$ are right ∡. | **2.** ⊥s form right ∡. |
| **3.** $\angle CAD$ is comp. $\angle 1$.<br>$\angle FBE$ is comp. $\angle 1$. | **3.** The acute ∡ of a right △ are complementary ∡. |
| **4.** ∴ $\angle CAD = \angle FBE$. | **4.** Compl. of the same $\angle$ are =. |
| Also, | |
| **5.** $\angle FBE$ is sup. $\angle FBI$. | **5.** Two ∡ are sup. if their ext. sides lie in the same st. line. |
| **6.** ∴ $\angle CAD$ sup. $\angle FBI$. | **6.** Sub. |

NOTE. Two ∡ whose sides are ⊥ right to right and left to left, are equal.

## Proposition XXIV. Theorem

**120.** *An exterior angle of a triangle is equal to the sum of its two remote interior angles.*



**Given** the ext. $\angle ACD$ of $\triangle ABC$.

**To prove** $\angle ACD = \angle A + \angle B$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $CE \parallel BA$. | 1. Through a pt. to draw a line ∥ to a given line. |
| 2. $\angle 1 = \angle A$. | 2. Alt. int. ∡ of ∥ lines are =. |
| 3. $\angle 2 = \angle B$. | 3. Corr. ∡ of ∥ lines are =. |
| 4. $\angle 1 + \angle 2 = \angle A + \angle B$. | 4. If = be added to = their sums are equal. |
| 5. $\angle ACD = \angle 1 + \angle 2$. | 5. Whole = sum of its parts. |
| 6. $\therefore \angle ACD = \angle A + \angle B$. | 6. Things = same thing are = to each other. |

---

Ex. **1.** Given two angles of a triangle, construct an angle equal to the remote exterior angle.

Ex. **2.** Given an exterior angle and a remote interior angle of a triangle. Construct an angle equal to the other remote interior angle.

Ex. **3.** If two angles of a triangle are equal, the bisector of the remote exterior angle is parallel to the opposite side of the triangle.

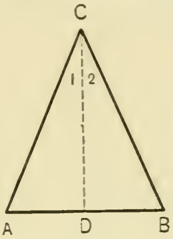
Ex. **4.** If at each vertex of a triangle, one exterior angle is drawn, the sum of the three angles is equal to four right angles.

Ex. **5.** The sum of two exterior angles of a triangle diminished by the third interior angle is equal to two right angles.

* Ex. **6.** If the sum of two exterior angles of a triangle is equal to three right angles, the triangle is a right one.

### PROPOSITION XXV. THEOREM

**121.** *If two angles of a triangle are equal, the sides opposite these angles are equal.* [Converse of Prop. IV.]



Given $\triangle ABC$, with $\angle A = \angle B$.

To prove $AC = BC$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw the bisector of $\angle C$ and extend it to meet $AB$ at $D$. | 1. Every $\angle$ has a bisector. |
| 2. $\angle A = \angle B$. | 2. Hyp. |
| 3. $\angle 1 = \angle 2$. | 3. Cons. |
| 4. $CD = CD$. | 4. Iden. |
| 5. $\triangle ACD \cong \triangle BCD$. | 5. s. a. a. = s. a. a. |
| 6. $\therefore AC = BC$. | 6. Corr. sides of $\cong$ $\triangle$ are =. |

**122.** COR. *An equiangular triangle is also equilateral.*

**123.** METHOD VII. **Propositions IV and XXV are occasionally used to prove the equality of lines or the equality of angles.** (Compare note on p. 31.)

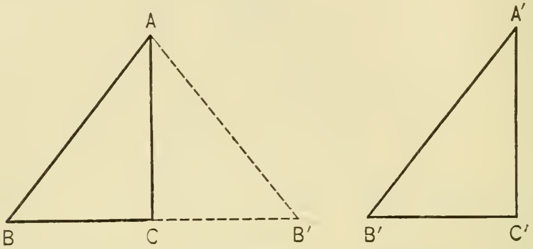
---

Ex. **1.** *If the vertex angle of an isosceles triangle is 60°, the triangle is equilateral.*

Ex. **2.** The bisectors of the base angles of an isosceles triangle, together with the base, form an isosceles triangle.

## PROPOSITION XXVI. THEOREM

**124.** *Two right triangles are congruent if the hypotenuse and an arm of the one are respectively equal to the hypotenuse and an arm of the other.* (*hy. arm = hy. arm.*)



**Given** $\triangle$s $ABC$ and $A'B'C'$, with $AB = A'B'$, $AC = A'C'$, and the rt. $\angle$s $C$ and $C'$.

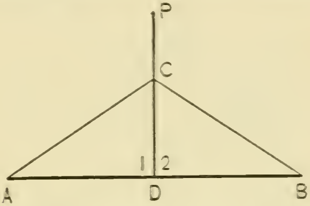
**To prove** $\triangle ABC \cong \triangle A'B'C'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Place $\triangle A'B'C'$ so that $A'C'$ coincides with $AC$, and that $B$ and $B'$ fall on opposite sides of $AC$. | **1.** Axiom 19.<br>Also, $AC = A'C'$, by hyp. |
| **2.** $\angle$s $BCA$ and $B'CA$ are rt. $\angle$s. | **2.** Hyp. |
| **3.** $\angle BCB'$ is a st. $\angle$. | **3.** A st. $\angle$ = sum of 2 right $\angle$s. |
| **4.** $BC$ and $CB'$ form a st. line $BCB'$. | **4.** Def. st. $\angle$. |
| **5.** $BCB'A$ is a $\triangle$, with $AB = AB'$. | **5.** $AB = A'B'$, by hyp. |
| **6.** $\angle B = \angle B'$. | **6.** The base $\angle$s of an isos. $\triangle$ are =. |
| **7.** $\therefore \triangle ABC \cong \triangle A'B'C'$. | **7.** (116). |

### PROPOSITION XXVII. THEOREM

**125.** *Every point in the perpendicular-bisector of a line is equidistant from the extremities of that line.*



**Given** $PD$, the perpendicular-bisector of $AB$, and $C$ any point in $PD$.

**To prove** $CA = CB$.

### Proof

| | STATEMENTS | REASONS |
|---|---|---|
| 1. | $AD = DB$. | 1. By hyp., $PD$ bisects $AB$. |
| 2. | $PD \perp AB$. | 2. Hyp. |
| 3. | $\angle 1 = \angle 2$. | 3. ⊥s form rt. ∠s.<br>Right ∠s are =. |
| 4. | $DC = DC$. | 4. Iden. |
| 5. | $\triangle ADC \cong \triangle BDC$. | 5. s. a. s. = s. a. s. |
| 6. | $\therefore CA = CB$. | 6. Corr. sides of ≅ △s are =. |

NOTE. The converse theorem is the corollary on p. 36.

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Ex. 1. In a given line $AB$ find a point equidistant from two given points $P$ and $Q$.

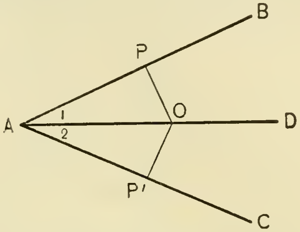
Ex. 2. In a given circle find a point equidistant from two given points.

Ex. 3. *Find a point equidistant from three given points.*

Ex. 4. Given three fixed points $A$, $B$, and $C$. Required to construct a point $X$, so that $AX = BX$, and $CX = \frac{1}{2}$ inch.

### PROPOSITION XXVIII. THEOREM

**126.** *Every point in the bisector of an angle is equidistant from the sides of the angle.*



**Given** $AD$ the bisector of $\angle BAC$, and $O$ any point in $AD$ and $OP \perp AB$, $OP' \perp AC$.

**To prove** $O$ is equidistant from $AB$ and $AC$ (*i.e.* $OP = OP'$).

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. $\angle 1 = \angle 2$. | 1. By hyp., $AD$ bis. $\angle BAC$. |
| 2. $OP \perp AB$, $OP' \perp AC$. | 2. Hyp. |
| 3. $\angle$s $APO$ and $AP'O$ are rt. $\angle$s. | 3. $\perp$s form rt. $\angle$s. |
| 4. $AO = AO$. | 4. Iden. |
| 5. $\triangle APO \cong \triangle AP'O$. | 5. Hy. $\angle$acute = hy. $\angle$acute. |
| 6. $\therefore OP = OP'$. | 6. Corr. sides of $\cong$ $\triangle$s are =. |

**127.** COR. *Every point equidistant from the sides of an angle is in the bisector of that angle.*

[Converse to Prop. XXVIII.]

HINT. Student should write out carefully the proof of this corollary. Prove $\triangle$s $APO$ and $AP'O$ are congruent and consequently $\angle 1 = \angle 2$.

---

Ex. **1.** In a given line $AB$ to find a point equidistant from the sides of the given $\angle DEF$.

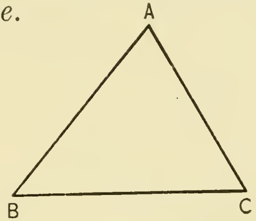
Ex. **2.** In a given circle find a point equidistant from two given lines.

Ex. **3.** *Find a point equidistant from three given lines.*

## UNEQUAL LINES AND UNEQUAL ANGLES

### PROPOSITION XXIX. THEOREM

**128.** *The sum of two sides of a triangle is greater than the third side.*



**Given** $\triangle ACB$.

**To prove** $AB + AC > BC$.

HINT. See Axiom 14.

---

EX. **1.** Is it possible to draw a triangle whose sides are

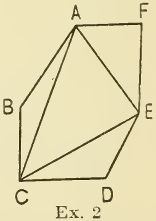
(*a*) 4 in., 5 in., 10 in.?

(*b*) 8 in., 19 in., 12 in.?

(*c*) 4 ft., 12 ft., 8 ft.?

(*d*) 6 in., 2 in., 5 in.?

EX. **2.** Prove that the perimeter of polygon $ABCDEF >$ perimeter of $\triangle ACE$.



Ex. 2

EX. **3.** If in $\triangle ABC$, $AC = BC$, and a point $D$ in the prolongation of $AC$ be connected with $B$, then $DA > DB$.

EX. **4.** If in the side $BC$ of $\triangle ABC$ any point $D$ be taken, then $AB + BC > AD + DC$.

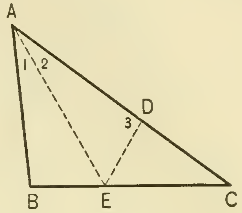
EX. **5.** If from any point $E$ in $\triangle ABC$, $EB$ and $EC$ are drawn, then $AB + AC > EB + EC$.

HINT. Produce $BE$ until it meets $AC$ in $D$.

EX. **6.** *The difference of any two sides of a triangle is less than the third side.*

## PROPOSITION XXX. THEOREM

**129.** *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



**Given** $\triangle ABC$ with $AC > AB$.

**To prove** $\angle B > \angle C$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw the angle-bisector $AE$. | **1.** Every ∠ has a bisector. |
| **2.** On $AC$ lay off $AD = AB$. | **2.** A line may be laid off on another line. |
| **3.** Draw $DE$. | **3.** Ax. 13. |
| **4.** $AB = AD$. | **4.** Con. |
| **5.** $AE = AE$. | **5.** Iden. |
| **6.** $\angle 1 = \angle 2$. | **6.** Con. |
| **7.** $\triangle ABE \cong \triangle ADE$. | **7.** s. a. s. = s. a. s. |
| **8.** $\angle B = \angle 3$. | **8.** Corr. ∡ of congruent ◬ are =. |
| **9.** $\angle 3 > \angle C$. | **9.** (88). |
| **10.** $\therefore \angle B > \angle C$. | **10.** Sub. |

---

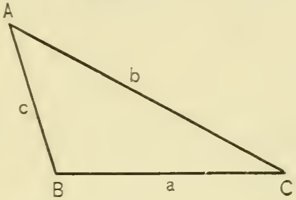
Ex. **1.** If one arm $AC$ of an isosceles triangle $ABC$ is produced to $D$, then $\angle ABD > \angle ADB$.

Ex. **2.** In $\triangle ABC$, $AB > AC$, and $\angle A = 60°$. Which is the greatest angle of the triangle?

### Proposition XXXI. Theorem

**130.** *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*

[Converse of Prop. XXX.]



**Given** $\triangle ABC$, with $\angle B > \angle C$.

**To prove** $b > c$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Either $b < c$, or $b = c$, or $b > c$. | **1.** Only 3 suppositions are possible. |
| **2.** If $b < c$, then $\angle B < \angle C$. | **2.** In a △, the greater ∠ lies opp. the greater side. |
| **3.** This is impossible. | **3.** By hyp., $\angle B > \angle C$. |
| **4.** If $b = c$, then $\angle B = \angle C$. | **4.** The base ∡ of an isos. △ are equal. |
| **5.** This is impossible. | **5.** By hyp., $\angle B > \angle C$. |
| **6.** $\therefore b > c$. | **6.** The other two suppositions are false. |

**131.** Cor. *The perpendicular is the shortest line that can be drawn from a point to a given line.* (118)

Note. The method used in the above proof is known as the **indirect method.** Instead of showing that a certain conclusion is true, we examine *all* conclusions which contradict the one to be proved, and demonstrate that these are false. Thus, in order to demonstrate $a = b$, we simply prove that the statement $a \neq b$ is false. Or to prove $m > n$, we disprove the only contradictory conclusions, viz. $m = n$, and $m < n$.

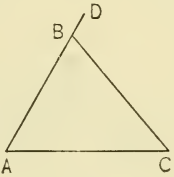
Ex. **1.** *Which is the greatest side of a right triangle? Of an obtuse triangle?*

Ex. **2.** If two angles of a triangle are 50° and 60° respectively, which is the greater of the two opposite sides? Which is the greatest side of the triangle? Which is the shortest?

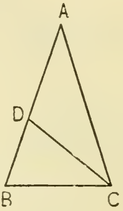
Ex. **3.** If in $\triangle ABC$, $AB > AC$ and $\angle B = 60°$, which is the greatest side of the triangle? Which is the smallest?

Ex. **4.** If in $\triangle ABC$, $AB > AC$, and $\angle A = 60°$, which is the greatest side of the triangle?

Ex. **5.** If $\angle A$ of $\triangle ABC$ equals two thirds of a right angle, and the exterior angle $DBC$ equals 110°, which is the greatest side of the triangle? Which is the shortest?



Ex. 5

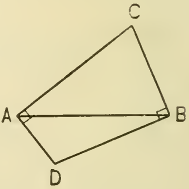


Ex. 6

Ex. **6.** If in $\triangle ABC$, $AB = AC$, prove that $DC > DB$.

Ex. **7.** In $\triangle ABC$, $AC > BC$. If the bisectors of angles $A$ and $B$ meet in $D$, prove $AD > BD$.

Ex. **8.** $AC > BC$, $AD \perp AC$, and $DB \perp BC$. Prove $BD > AD$.



Ex. 8

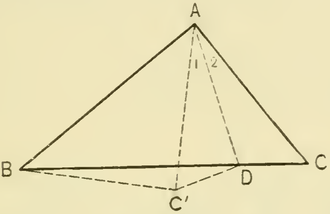
Ex. **9.** Prove Prop. XXXI directly.

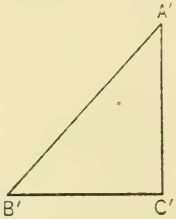
Hint. At $B$ draw $\angle CBD = \angle C$.

Ex. **10.** The sum of the altitudes of a triangle is less than the perimeter of the triangle.

## PROPOSITION XXXII. THEOREM

**132.** *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*





**Given** $\triangle ABC$ and $A'B'C'$ with

$$AB = A'B',\ AC = A'C',\ \angle A > \angle A'.$$

**To prove** $BC > B'C'$.

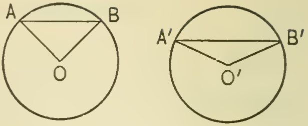
### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Apply $\triangle A'B'C'$ to $\triangle ABC$ so that $A'B'$ coincides with $AB$. | 1. By hyp., $AB = A'B'$ and Ax. 19. |
| 2. $A'C'$ will fall between $AB$ and $AC$ as $AC'$. | 2. By hyp., $\angle A > A'$. |
| 3. Draw $AD$ bisecting $\angle CAC'$ to meet $BC$ in $D$, and draw $DC'$. | 3. Every $\angle$ has a bisector.<br>To draw a st. line. |
| 4. $AC' = AC$. | 4. Hyp. |
| 5. $\angle 1 = \angle 2$. | 5. Con. |
| 6. $AD = AD$. | 6. Iden. |
| 7. $\triangle C'AD \cong \triangle CAD$. | 7. s. a. s. = s. a. s. |
| 8. $DC' = DC$. | 8. Why? |
| 9. $BD + DC' > BC'$. | 9. (128). |
| 10. $BD + DC > BC'$. | 10. Sub. |
| 11. $\therefore BC > BC'$ or $B'C'$. | 11. Sub. |

**Ex. 1.** If $O$ and $O'$ are two equal circles, and

$$\angle A'O'B' > \angle AOB,$$

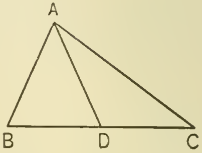
then $$A'B' > AB.$$



Ex. 1

**Ex. 2.** If in triangle $ACB$ the median $AD$ makes $\angle ADB$ an acute $\angle$, prove

(a) $AC > AB$.
(b) $\angle B > \angle C$.



Ex. 2

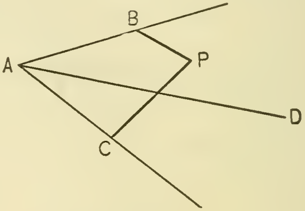
**Ex. 3.** If in $\triangle ABC$, $\angle A = \angle B$, and a point $D$ in $AB$ be taken so that $\angle ACD > \angle DCB$, then $AD > DB$.

**Ex. 4.** *Every point not in the perpendicular bisector of a line is unequally distant from the ends of the line.*

Hint. Apply Prop. XXXII.

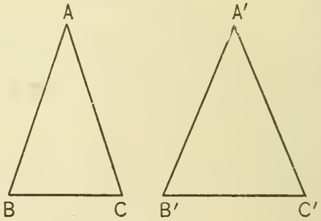
**Ex. 5.** $AD$ is the bisector of $\angle BAC$ and $AB = AC$. Prove

$$PC > PB.$$



Ex. 5

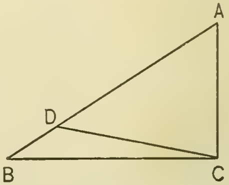
**Ex. 6.** $\triangle ABC$ and $\triangle A'B'C'$ are isosceles $\triangle$s with bases $BC$ and $B'C'$. If $AB = A'C'$ and $\angle B > \angle B'$, which $\triangle$ has the greater base?



Ex. 6

**Ex. 7.** Prove Prop. XXXII when point $C'$ falls inside $\triangle ABC$.

**Ex. 8.** In $\triangle BCA$, $AC \perp BC$ and $AD = BC$. Prove $AB > DC$.

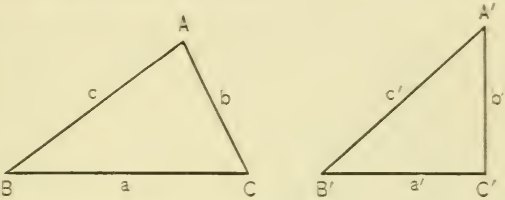


Ex. 8

### PROPOSITION XXXIII. THEOREM

**133.** *If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

[Converse of Prop. XXXII.]



**Given** $\triangle ABC$ and $A'B'C'$, with $b = b'$, $c = c'$, and $a > a'$.

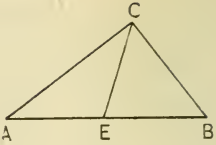
**To prove** $\angle A > \angle A'$.

Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Either $\angle A < \angle A'$, or $\angle A = \angle A'$, or $\angle A > \angle A'$. | 1. Only 3 suppositions are possible. |
| **2.** If $\angle A < \angle A'$, then $a < a'$. | 2. If two $\triangle$ have 2 sides of the one equal respectively to 2 sides of the other, but the included $\angle$ of the first > the included $\angle$ of the second, then the 3d side of the first > 3d side of the second. |
| **3.** This is impossible. | 3. $a > a'$, by hyp. |
| **4.** If $\angle A = \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$. | 4. s. a. s. = s. a. s. |
| **5.** $a = a'$. | 5. Corr. sides, etc. |
| **6.** This is impossible. | 6. $a > a'$, by hyp. |
| **7.** $\therefore \angle A > \angle A'$. | 7. The other two suppositions have been proved false. |

Ex. **1**. If in $\triangle ABC$, having $AC > BC$, the median $CE$ is drawn, then $\angle AEC$ is an obtuse angle.

Ex. **2**. Given $\triangle ABC$, having $\angle A = \angle B$. If the point $D$ in $AB$ be taken so that $AD > DB$, then $\angle ACD > \angle DCB$.



Ex. 1

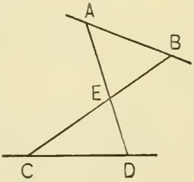
**134.** METHOD VIII. **Inequality of lines is generally proved by means of Props. XXIX, XXXI, XXXII. If it is impossible to discover any relation of the angles, Prop. XXIX is used; if the two lines are parts of the same triangle, and the opposite angles can be proved unequal, Prop. XXXI is used; but if the sides or angles are parts of different triangles, Prop. XXXII is used. In some cases several methods will lead to the desired result.**

**The inequality of angles is proved in a similar manner by Props. XII, XXX, and XXXIII.**

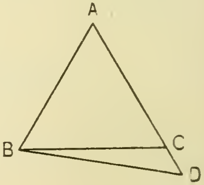
---

Ex. **3**. If $AD$ and $BC$ intersect in $E$, prove $AD + BC > AB + CD$. (Which of the three propositions must be used?)

Ex. **4**. If $AC$, a side of an equilateral triangle, is produced to $D$, and $BD$ is joined, then $AD > BD > AB$.



Ex. 3



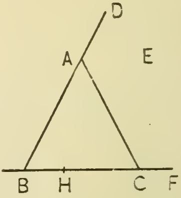
Ex. 4

Ex. **5**. If $AB = AC$, prove that

(*a*) $BD > DC$. (*c*) $AF > AB$.
(*b*) $BE > EC$. (*d*) $AB > AH$.

Ex. **6**. The sum of the diagonals of any quadrilateral is less than the perimeter, but greater than the semiperimeter of the quadrilateral.

* Ex. **7**. If in $\triangle ABC$, having $AB < BC$, $BD$, the bisector of angle $B$, is drawn, meeting $AC$ in $D$, then $AD < DC$.

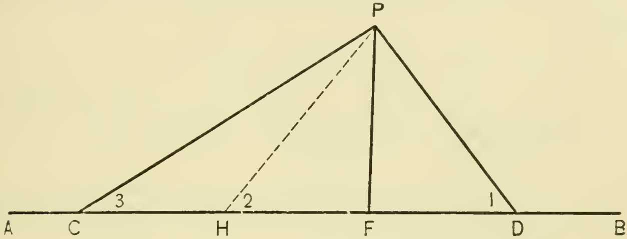


Ex. 5

* Ex. **8**. If in $\triangle ABC$, having $AC > BC$, the median $CD$ is drawn, then any point $E$ in $CD$ is nearer to $B$ than to $A$.

### PROPOSITION XXXIV. THEOREM

**135.** *If, from a point in a perpendicular to a given line, two oblique lines are drawn to the line, the one cutting off the greater distance from the foot of the perpendicular is the longer line.*



**Given** line $AB$, point $P$, $PF \perp AB$, $CF > FD$.

**To prove** $PC > PD$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** On $FC$ lay off $FH = FD$. | **1.** To lay off a given length on a line. |
| **2.** Draw $HP$. | **2.** A st. line can be drawn between two pts. |
| **3.** $PH = PD$. | **3.** Every pt. in the $\perp$ bisector of a line is equidistant from the ends of the line. |
| **4.** $\angle 2 = \angle 1$. | **4.** If two sides of a △ are =, the ∡ opp. are =. |
| **5.** $\angle 2 > \angle 3$. | **5.** An ext. ∠ of a △ > either remote int. ∠. |
| **6.** $\angle 1 > \angle 3$. | **6.** Sub. |
| **7.** $\therefore PC > PD$. | **7.** If two ∡ of a △ are unequal, opp. the greater side lies the greater ∠. |

---

Ex. **1.** *State and prove the converse of Prop. XXXIV.*

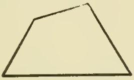
Ex. **2.** *Prove that from a given point outside a given line only two equal oblique lines can be drawn to the given line.*

## QUADRILATERALS

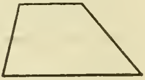
136. Def. A **quadrilateral** is a polygon of four sides.

A **trapezoid** is a quadrilateral having two, and only two, sides parallel.

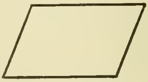
A **parallelogram** is a quadrilateral having its opposite sides parallel.



Quadrilateral



Trapezoid

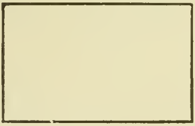


Parallelogram

137. Def. A **rectangle** is a parallelogram whose angles are right angles.

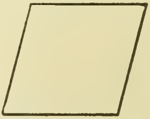
A **square** is an equilateral rectangle.

A **rhombus** is an equilateral parallelogram.



Rectangle

Square



Rhombus

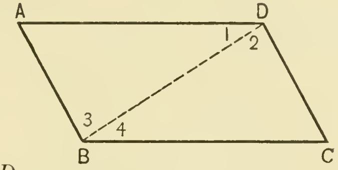
138. Def. An **isosceles trapezoid** is a trapezoid whose nonparallel sides are equal. The parallel sides of a trapezoid are called its **bases**, and are distinguished as **upper** and **lower**.

The **median** of a trapezoid is the line joining the mid-points of the nonparallel sides.

A **diagonal** of a quadrilateral is a straight line joining opposite vertices. The **altitude** of a parallelogram or trapezoid is the perpendicular distance between the two bases.

### PROPOSITION XXXV. THEOREM

**139.** *The opposite sides and the opposite angles of a parallelogram are equal.*



**Given** $\square ABCD$.

**To prove** $AD = BC$, $AB = CD$, $\angle A = \angle C$, $\angle B = \angle D$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $BD$. | 1. A st. line can be drawn between two pts. |
| 2. $AD \parallel BC$, $AB \parallel DC$. | 2. The opp. sides of a $\square$ are $\parallel$. |
| 3. $\angle 1 = \angle 4$, $\angle 3 = \angle 2$. | 3. Alt. int. $\angle$s of $\parallel$ lines are =. |
| 4. $BD = BD$. | 4. Iden. |
| 5. $\triangle ABD \cong \triangle BDC$. | 5. a. s. a. = a. s. a. |
| 6. $AD = BC$.<br>$AB = CD$.<br>$\angle A = \angle C$. | 6. Corr. parts of $\cong$ $\triangle$s are =. |
| 7. In like manner by drawing $AC$, we can prove $\angle B = \angle D$. | 7. Steps similar to 1–6. |

**140.** COR. 1. *A diagonal divides a parallelogram into two congruent triangles.*

**141.** COR. 2. *Parallels included between parallels are equal.*

**142.** COR. 3. *Parallel lines are everywhere equidistant.*

---

EX. **1.** *The diagonals of a rectangle are equal.*

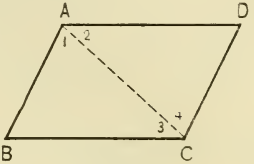
EX. **2.** *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

EX. **3.** The perpendiculars to a diagonal of a parallelogram from the opposite vertices are equal.

### PROPOSITION XXXVI. THEOREM

**143.** *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

[Converse of Prop. 35.]



**Given** quadrilateral $ABCD$ with

$$AB = CD,\ AD = BC.$$

**To prove** $ABCD$ is a ▱.

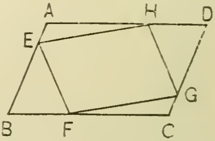
### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $AC$. | **1.** A st. line can be drawn between two points. |
| **2.** $AB = CD$, $AD = BC$. | **2.** Hyp. |
| **3.** $AC = AC$. | **3.** Iden. |
| **4.** $\triangle ABC \cong \triangle ACD$. | **4.** s. s. s. = s. s. s. |
| **5.** $\angle 1 = \angle 4$, $\angle 3 = \angle 2$. | **5.** Corr. $\angle$s of congruent $\triangle$s are =. |
| **6.** $AB \parallel CD$, $BC \parallel AD$. | **6.** If a pair of alt. int. $\angle$s are =, the lines are ∥. |
| **7.** ∴ $ABCD$ is a ▱. | **7.** Its opp. sides are parallel. |

---

Ex. **1.** If $ABCD$ is a parallelogram, $AE = CG$, and $BF = DH$, then $EFGH$ is a parallelogram.

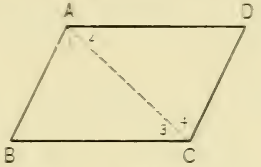
Ex. **2.** The lines joining the mid-points of the sides of a parallelogram, taken in order, form a parallelogram.



Ex. 1

### PROPOSITION XXXVII. THEOREM

**144.** *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



**Given** quadrilateral $ABCD$ with

$$AD \parallel BC, AD = BC.$$

**To prove** $ABCD$ is a ▱.

Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $AC$. | 1. Ax. 13. |
| 2. $AD \parallel BC$. | 2. Hyp. |
| 3. $\angle 2 = \angle 3$. | 3. Alt. int. $\angle$s of $\parallel$ lines are =. |
| 4. $AD = BC$. | 4. Hyp. |
| 5. $AC = AC$. | 5. Iden. |
| 6. $\triangle ADC \cong \triangle ABC$. | 6. s. a. s. = s. a. s. |
| 7. $\angle 4 = \angle 1$. | 7. Corr. $\angle$s of $\cong$ $\triangle$s are =. |
| 8. $BA \parallel CD$. | 8. (93). |
| 9. $\therefore ABCD$ is a ▱. | 9. Both pairs of opp. sides are $\parallel$. |

Ex. 1. If on opposite sides of $AB$ the parallelograms $ABCD$ and $ABEF$ are drawn, then $CDFE$ is also a parallelogram.

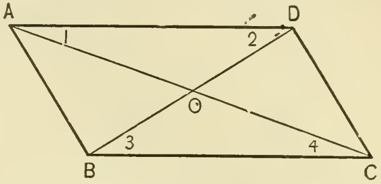
Ex. 2. If two opposite sides of a parallelogram are produced by the same length in opposite directions and their ends joined to the nearest vertices, another parallelogram is formed.

**145.** METHOD IX. **Lines may be shown to be parallel by proving them to be opposite sides of a parallelogram.**

Ex. 3. If two opposite sides of a parallelogram are divided into three equal parts, and the respective points of division are joined, the lines are parallel.

## PROPOSITION XXXVIII. THEOREM

**146.** *The diagonals of a parallelogram bisect each other.*



**Given** ▱ $ABCD$ having diagonals $AC$ and $BD$.

**To prove** $AO = OC,\ BO = OD$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** $AD \parallel BC,\ AB \parallel DC$. | **1.** By hyp., the fig. is a ▱. |
| **2.** $\angle 1 = \angle 4,\ \angle 2 = \angle 3$. | **2.** Alt. int. ∠s of $\parallel$ lines are =. |
| **3.** $AD = BC$. | **3.** Opp. sides of ▱ are =. |
| **4.** $\triangle AOD \cong \triangle BOC$. | **4.** a. s. a. = a. s. a. |
| **5.** $\therefore AO = OC,\ BO = OD$. | **5.** Corr. sides of $\cong$ △s are =. |

---

Ex. **1.** *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.* [Converse to Prop. 38.]

Ex. **2.** *The diagonals of a rhombus are perpendicular to each other.*

Ex. **3.** *If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus.*

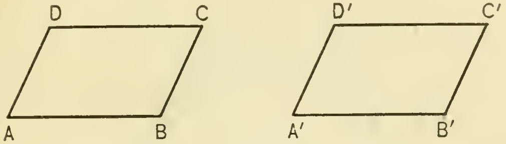
Ex. **4.** If each half of the diagonals of a parallelogram is bisected and the mid-points are joined in order, another parallelogram is formed.

Ex. **5.** If a diagonal bisects an angle of a parallelogram, the figure is a rhombus.

Ex. **6.** If the ends of two diameters of a circle be joined in succession, a rectangle is formed.

### PROPOSITION XXXIX. THEOREM

**147.** *Two parallelograms are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*



**Given** ▱s $ABCD$ and $A'B'C'D'$ with $AB = A'B'$, $AD = A'D'$, $\angle A = \angle A'$.

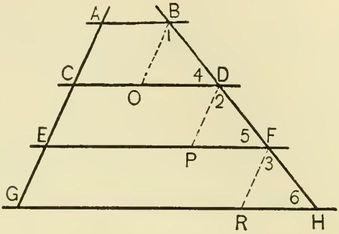
**To prove** ▱ $ABCD \cong$ ▱ $A'B'C'D'$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Place ▱$ABCD$ upon $A'B'C'D'$, so that $AB$ coincides with $A'B'$. | **1.** Axiom 19.<br>By hyp., $AB = A'B'$. |
| **2.** Then $AD$ falls along $A'D'$. | **2.** By hyp., $\angle A = \angle A'$. |
| **3.** $D$ coincides with $D'$. | **3.** By hyp., $AD = A'D'$. |
| **4.** $BC$ falls along $B'C'$. | **4.** Axiom 16. |
| **5.** $C$ must lie in $B'C'$ or in $B'C'$ produced. | **5.** Step 4. |
| **6.** Similarly, $C$ must lie in $D'C'$ or in $D'C'$ produced. | **6.** Axiom 16. |
| **7.** Hence $C$ coincides with $C'$, and ▱$ABCD$ coincides with ▱$A'B'C'D'$. | **7.** Axiom 13 *a*. |
| **8.** $\therefore$ ▱$ABCD \cong$ ▱$A'B'C'D'$. | **8.** Figures are $\cong$ if they can be made to coincide. |

### PROPOSITION XL. THEOREM

**148.** *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



**Given** $AB \parallel CD \parallel EF \parallel GH$, and $AC = CE = EG$.

**To prove** $BD = DF = FH$.

**Proof**

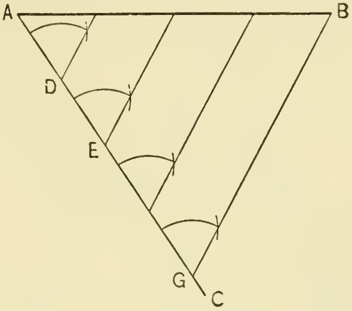
| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $BO$, $DP$, and $FR \parallel AG$. | **1.** (100). |
| **2.** $BO \parallel DP \parallel FR$. | **2.** Lines $\parallel$ to the same line are $\parallel$ to each other. |
| **3.** $\begin{cases} \angle 1 = \angle 2 = \angle 3. \\ \angle 4 = \angle 5 = \angle 6. \end{cases}$ | **3.** Corr. ∡ of $\parallel$ lines are =. |
| **4.** $BO = AC$, $DP = CE$, $FR = EG$. | **4.** (141). |
| **5.** $AC = CE = EG$. | **5.** Hyp. |
| **6.** $BO = DP = FR$. | **6.** Ax. 1. |
| **7.** $\triangle BOD \cong \triangle DPF \cong \triangle FRH$. | **7.** s. a. a. = s. a. a. |
| **8.** $\therefore BD = DF = FH$. | **8.** Corr. sides of $\cong$ △ are =. |

**149.** COR. 1. *The line parallel to one side of a triangle, bisecting another side, bisects the third side.*

**150.** COR. 2. *The line that bisects a nonparallel side of a trapezoid and is parallel to the bases bisects the other nonparallel side.*

### PROPOSITION XLI. PROBLEM

**151.** *To divide a given line into any number of equal parts.*



**Given** line $AB$.

**Required** to divide $AB$ into $n$ equal parts.

**Construction.** From $A$ draw any line $AC$.

On $AC$ lay off any segment, as $AD$, $n$ times in succession, as $AD$, $DE$, etc. Join the last point of division, $G$, with $B$, and through the other points of division draw ||s to $GB$.

NOTE. Proof: Through $A$ draw || to $GB$, then apply Prop. 40.

EX. Divide a line into three equal parts. Into five equal parts.

**152.** METHOD X. **To prove that a line is twice as large as another, we usually double the smaller and prove that its double equals the greater, or sometimes we bisect the greater and prove that its half equals the smaller. The same relation between angles is proved in a similar way.**

---

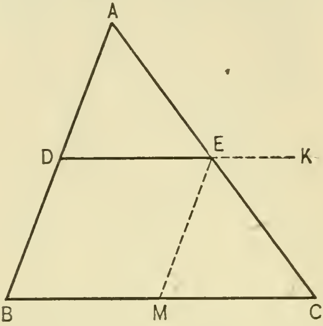
EX. **1.** If two angles of a triangle are 30° and 60° respectively, the side opposite the angle of 30° is one half the side included by the two angles.

EX. **2.** In triangle $ABC$, if through the mid-point $D$ of $AC$ a line $DE$ is drawn parallel to $AB$, and $E$ lies in $CB$, then $DE$ is one half $AB$.

EX. **3.** The angle formed by the base of an isosceles triangle and the altitude upon one arm is one half the vertical angle.

### Proposition XLII. Theorem

**153.** *A line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*



**Given** $\triangle ABC$ with $AD = DB$, $AE = EC$.
**To prove** $DE \parallel BC$, and $DE = \frac{1}{2} BC$.

**Proof**

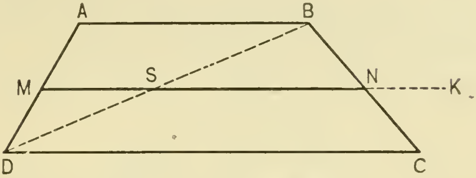
| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $DK \parallel BC$. | **1.** (100). |
| **2.** $DK$ bisects $AC$ (*i.e.* passes through point $E$). | **2.** Prop. 40, Cor. 1. |
| **3.** $DE$ coincides with $DK$. | **3.** Ax. 13. |
| **4.** $\therefore DE \parallel BC$. | **4.** It is the same line as $DK$. |
| **5.** Draw $EM \parallel AB$. | **5.** Same as 1. |
| **6.** $BM = MC$. | **6.** Same as 2. |
| **7.** $DE = BM$. | **7.** Prop. 35, Cor. 2. |
| **8.** $\therefore DE = \frac{1}{2}(BC)$. | **8.** By 6, 7, $DE = BM$, which is one half of $BC$. |

---

Ex. **1.** If the mid-points of the three sides of a triangle are joined, four congruent triangles are formed.

Ex. **2.** A line joining the mid-points of two adjacent sides of a quadrilateral is equal and parallel to the line joining the mid-points of the other two.

Ex. **3**. *The median of a trapezoid is parallel to the bases, and equal to half the sum of the bases.*



Given trapezoid $ABCD$ with median $MN$.

To prove $\begin{cases} MN \parallel DC \parallel AB. \\ MN = \frac{1}{2}(AB + DC). \end{cases}$

## Proof

| STATEMENTS | REASONS |
|---|---|
| **1**. Draw $MK \parallel DC$. It is $\parallel$ to $AB$. | **1**. Why? |
| **2**. $MK$ passes through $N$. | **2**. Prop. 40. |
| **3**. $MN \parallel DC$, also $\parallel AB$. | **3**. Why? |
| **4**. Draw diag. $BD$. | **4**. Why? |
| **5**. $S$ is mid-point of $DB$. | **5**. Prop. 40, Cor. 1. |
| **6**. $MS = \frac{1}{2} AB$, $SN = \frac{1}{2} DC$. | **6**. Prop. 42. |
| **7**. $MN = \frac{1}{2}(AB + DC)$. | **7**. Why? |

---

Ex. **4**. If the mid-points of the arms of an isosceles triangle are joined to the mid-points of the base, a rhombus is formed.

**154**. Def. An **equiangular polygon** is a polygon having all of its angles equal. An **equilateral polygon** is a polygon having all of its sides equal.

A polygon of five sides is called a **pentagon**; six sides, a **hexagon**; seven sides, a **heptagon**; eight sides, an **octagon**; ten sides, a **decagon**; twelve sides, a **dodecagon**.

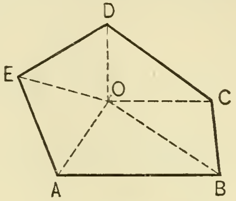
**155**. *Polygons that are mutually equiangular and equilateral are* **congruent**, *for they can be made to coincide.*

Ex. **5**. Draw two mutually equiangular quadrilaterals that are not mutually equilateral.

Ex. **6**. Draw two mutually equilateral quadrilaterals that are not mutually equiangular.

### PROPOSITION XLIII. THEOREM

**156.** *The sum of all the angles of a polygon of n sides is equal to (n − 2) straight angles.*



**Given** *ABCDE* ⋯, a polygon of $n$ sides.

**To prove** $\angle A + \angle B + \angle C + \angle D + \cdots = (n-2)$ st. ∡.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Connect any point within, as $O$, with all the vertices. | 1. Why? |
| 2. $n$ △ are formed. | 2. Each side of polygon is a base. |
| 3. Sum of ∡ of poly. = sum all base ∡ of all △. | 3. Ax. 10. |
| 4. Sum of all ∡ of △ = $n$ st. ∡. | 4. Why? |
| 5. Sum of vertex ∡ of △ = 2 st. ∡. | 5. Why? |
| 6. Sum of base ∡ of △ = $(n-2)$ st. ∡. | 6. Why? |
| 7. ∴ Sum all ∡ of poly. = $(n-2)$ st. ∡. | 7. Why? |

*i.e.* $\angle A + \angle B + \angle C + \cdots = (n-2)$ st. ∡.

**157.** COR. *Each angle of an equiangular polygon of n sides equals* $\frac{n-2}{n}$ *of a straight angle.*

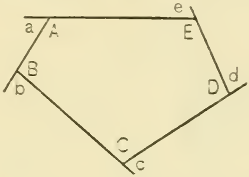
---

Ex. **1.** How many sides has a polygon the sum of whose angles is 1000 st. ∡? 200 rt. ∡? 24 rt. ∡? 720°?

Ex. **2.** How many degrees are in each angle of an equiangular quadrilateral? Pentagon? Hexagon? Decagon?

### PROPOSITION XLIV. THEOREM

**158.** *If the sides of any polygon be successively produced, forming one exterior angle at each vertex, the sum of these angles is equal to two straight angles.*



**Given** $ABCDE$, a polygon of $n$ sides, with the ext. ∡ $a$, $b$, $c$, etc.

**To prove** $\angle a + \angle b + \angle c + \cdots = 2$ st. ∡.

**Proof.** $\angle A + \angle a = 1$ st. ∠, $\angle B + \angle b = 1$ st. ∠, etc. (50)

$\therefore (\angle A + \angle B + \cdots) + (\angle a + \angle b + \cdots) = n$ st. ∡. (Ax. 2.

But $(\angle A + \angle B + \cdots) = (n-2)$ st. ∡. (156

$\therefore \angle a + \angle b + \angle c + \cdots = 2$ st. ∡. (Ax. 3.)

**159.** COR. *Each exterior angle of an equiangular polygon of $n$ sides equals the $\frac{2}{n}$ part of a st. ∠.*

---

Ex. **1.** How many degrees are in each exterior angle of an equiangular polygon of 10 sides? Of 9 sides? Of 36 sides? Of 72 sides?

Ex. **2.** How many sides has a polygon each of whose exterior angles equals 30°? One right angle? 60°? 45°?

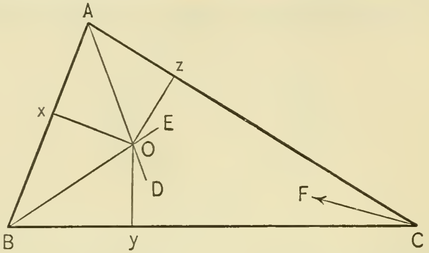
Ex. **3.** How many sides has a polygon each of whose interior angles equals 160°? 179°? 135°? $\frac{4}{3}$ rt. ∡?

Ex. **4.** How many sides has an equiangular polygon, three of whose exterior angles are together equal to 90°? $\frac{3}{4}$ of a right ∠?

Ex. **5.** How many sides has a polygon, the sum of whose interior angles equals twice the sum of its exterior angles?

### PROPOSITION XLV. THEOREM

**160.** *The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.*



**Given** $\triangle ABC$, and $AD$, $BE$, and $CF$, the bisectors of $\angle$s $A$, $B$, and $C$ respectively.

**To prove** (*a*) $AD$, $BE$, and $CF$ meet in a point, as $O$.
(*b*) $O$ is equidistant from $AB$, $BC$, and $CA$.

**Proof.** Since $AD$ and $BE$ cannot be $\parallel$, they meet in a point, as $O$. (107)

From $O$ draw $OX$, $OY$, and $OZ$ respectively $\perp$ $AB$, $BC$, and $CA$.

$$OX = OZ. \qquad (126)$$

$$OX = OY. \qquad (126)$$

$$\therefore OZ = OY. \qquad (\text{Ax. } 1.)$$

$\therefore$ $O$ lies in $CF$ or its prolongation. (127)

$\therefore$ $AD$, $BE$, and $CF$ meet in point $O$, and $O$ is equidistant from $AB$, $BC$, and $CA$.

**161.** DEF. Three or more lines are **concurrent** if they meet in a common point.

---

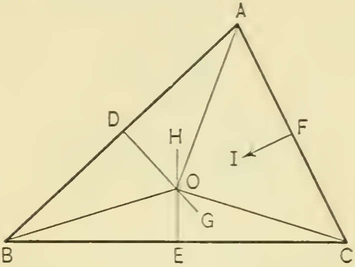
Ex. **1.** *Find by a construction a point equidistant from the three sides of a triangle.*

Ex. **2.** Prove that the bisectors of two exterior angles and the bisector of the third interior angle of a triangle meet in a point.

Ex. **3.** Construct a point equidistant from the sides $AB$, $BC$, and $CD$ of parallelogram $ABCD$.

### PROPOSITION XLVI. THEOREM

**162.** *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.*



**Given** $\triangle ABC$, and $DG$, $EH$, and $FI$ the perpendicular-bisectors of $AB$, $BC$, and $CA$ respectively.

**To prove** (*a*) $DG$, $EH$, and $FI$ meet in a common point. (*b*) The point of intersection is equidistant from $A$, $B$, and $C$.

**Proof.** $DG$ and $EH$ intersect at a point, as $O$. (102)

$OA = OB$ and $OB = OC$. (125)

$\therefore OA = OC$. (Ax. 1.)

$\therefore O$ lies in the $\perp$ bisector of $AC$. (80)

Or $DG$, $EH$, and $FI$ meet in $O$; and $O$ is equidistant from $A$, $B$, and $C$ since $OA = OB = OC$.

---

Ex. **1.** Construct a point $O$ equidistant from three vertices of a quadrilateral.

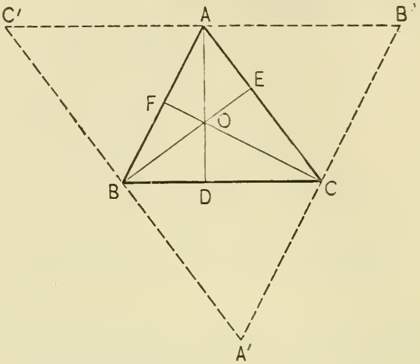
Ex. **2.** *Construct a circle which passes through the three vertices of a given triangle.*

Ex. **3.** Prove that only *one* circle can be passed through the three vertices of a triangle.

Ex. **4.** Prove that the center of the circle about an equilateral $\triangle$ lies in each of its $\angle$ bisectors.

### Proposition XLVII. Theorem

**163.** *The three altitudes of a triangle (produced if necessary) meet in a point.*



**Given** $\triangle ABC$ and the altitudes $AD$, $BE$, and $CF$.

**To prove** $AD$, $BE$, and $CF$ meet in a common point.

**Proof.** Through the vertices of $ABC$, draw $B'C' \parallel BC$, $A'C' \parallel AC$, and $A'B' \parallel AB$. (Why?)

Since $AD \perp BC$, (Hyp.)

$AD \perp B'C'$. (105)

But $ABCB'$ and $ACBC'$ are parallelograms. (136)

$\therefore AB' = BC$, and $AC' = BC$. (139)

$\therefore AB' = AC'$. (Ax. 1.)

Or $AD$ is the $\perp$ bisector of $B'C'$.

In like manner it follows that

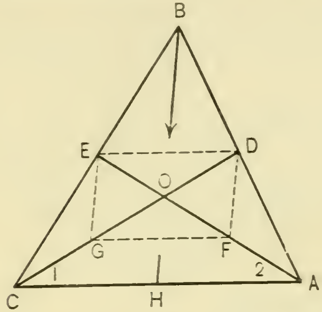
$BE$ is the $\perp$ bisector of $A'C'$,

and $CF$ is the $\perp$ bisector of $A'B'$.

$\therefore AD$, $BE$. and $CF$ meet in a point (*i.e.* are concurrent). (162)

## Proposition XLVIII. Theorem

**161.** *The medians of a triangle intersect in a point which is two thirds of the distance from a vertex to the mid-point of the opposite side.*



**Given** $\triangle ABC$ with medians $AE$, $CD$, $BH$.

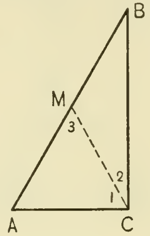
**To prove** $AE$, $CD$, and $BH$ meet in a point $O$, and $AO = \frac{2}{3}AE$, $CO = \frac{2}{3}CD$, $BO = \frac{2}{3}BH$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Let $CG = GO$ and $OF = FA$. | **1.** (81) |
| **2.** Draw $ED$, $DF$, $FG$, and $GE$. | **2.** Why? |
| **3.** $ED = \frac{1}{2}CA$ and $ED \parallel CA$. | **3.** (153) |
| **4.** $GF = \frac{1}{2}CA$ and $GF \parallel CA$. | **4.** (153) |
| **5.** $ED = GF$ and $ED \parallel GF$. | **5.** Why? |
| **6.** $EGFD$ is a parallelogram. | **6.** (144) |
| **7.** $OE = OF$ and $OD = OG$. | **7.** (146) |
| **8.** Hence $OE = OF = FA$ and $OD = OG = CG$. | **8.** Why? |
| **9.** $AO = \frac{2}{3}AE$, and $CO = \frac{2}{3}CD$. | **9.** Why? |
| **10.** Likewise $BH$ passes through $O$ and $BO = \frac{2}{3}BH$. | **10.** Steps (1–9) |

## Proposition XLIX. Theorem

**165.** *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter side (arm).*



**Given** right $\triangle ABC$, right $\angle$ at $C$, and $\angle A = 2\angle B$.

**To prove** $AB = 2\,AC$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. $\angle A = 2\angle B$, $\angle C = 90°$. | 1. Hyp. |
| 2. $\angle A + \angle B = 90°$. | 2. Why? |
| 3. $\therefore\ \angle A = 60°$, $\angle B = 30°$. | 3. Why? |
| 4. Draw $CM$ making $\angle 1 = \angle A = 60°$. | 4. Why? |
| 5. $\angle 3 = 60°$. | 5. Why? |
| 6. $\angle 2 = 30°$. | 6. Why? |
| 7. $AM = AC = MC$. | 7. Why? |
| 8. $MB = MC$. | 8. Why? |
| 9. $\therefore AB = AM + MB = 2\,AC$. | 9. Why? |

---

Ex. **1.** If in triangle $ABC$, $\angle A = 60°$, $\angle B = 30°$, and $AB = 12$ inches, find the length of $AC$.

Ex. **2.** If in triangle $ABC$, $AB = 10$ inches, $\angle A = 60°$, $\angle C = 90°$, and $CD$ is the altitude upon $AB$, find the length of $AD$.

Ex. **3.** The base angles $A$ and $B$ of isosceles triangle $ABC$ equal 15°, and $BC = 8$ inches. Find the length of the altitude $AD$.

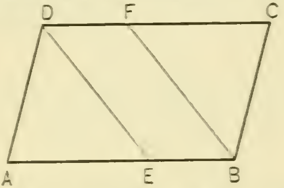
Ex. **4.** In quadrilateral $ABCD$, $AB = BC = 6$ inches, $\angle A = 120°$, $\angle B = 60°$, and $\angle C = 150°$. Find the length of $AD$.

Ex. **5.** $DC$, the upper base of trapezoid $ABCD$, equals 14 inches, and $AB = BC$. Find the length of $AB$, if $\angle A = 90°$, and $\angle B = 60°$.

## ANALYSIS OF THEOREMS

**166.** The demonstration of a theorem is most frequently discovered by the so-called **analytic method.** An analysis considers the various "*means*" by which a conclusion can be proved, and thereby reduces the original proposition to another, simpler one. This reduction is continued until a known fact is obtained. A concrete example will explain the true character of this method.

**167.** THEOREM. *The bisectors of a pair of opposite angles of an oblique parallelogram are parallel.*



**Given** $DE$ and $BF$, the bisectors of two angles of the oblique parallelogram $ABCD$.

**To prove** $DE \parallel BF$.

**Analysis.** 1. The means for proving that two lines are parallel are:

(*a*) A parallelogram.
(*b*) Equal alternate interior angles.
(*c*) Corresponding angles.
(*d*) Supplementary interior angles, etc.

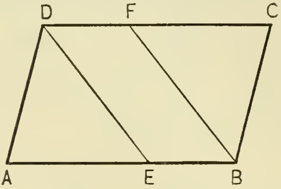
Let us select any one of these methods, *e.g.* 1 (*a*), *i.e.*
**to prove** $EBFD$ is a parallelogram.

2. The means for proving that a quadrilateral is a ▱ are:
(*a*) Opposite sides are equal.
(*b*) Two opposite sides are equal and parallel, etc.

Again we select any one, *e.g.* 2 (*a*), *i.e.*
**to prove** $ED = BF$, $EB = DF$.

3. The means for proving the equality of lines is usually a pair of congruent triangles, *i.e.*

**to prove** $\triangle AED \cong \triangle BCF$.



4. The congruence of the two triangles is easily established. Hence we obtain the following demonstration:

**Proof.** $AD = BC$. (139)

$\angle A = \angle C$. (139)

$\angle ADE = \angle FBC$. (Ax. 8.)

$\therefore \triangle ADE \cong \triangle FBC$. (a. s. a. = a. s. a.)

$\therefore ED = BF$, and $AE = FC$. (corr. parts, etc.)

But $AB = DC$. (139)

$\therefore DF = EB$. (Ax. 3.)

$\therefore DEBF$ is a ▱. (143)

$\therefore ED \parallel BF$. (136)

The above demonstration is not the only one, nor the shortest one. Each of the possibilities indicated under 1 (*a*), 1 (*b*), 1 (*c*), 1 (*d*), 2 (*a*), 2(*b*), will furnish one or more proofs.

---

Ex. **1.** Prove the above proposition by means of 2 (*b*).

Ex. **2.** Prove the above proposition by means of 1 (*b*).

Ex. **3.** Prove the above proposition by means of 1 (*c*).

**168.** When in an analysis new lines or angles are drawn, the student should — before proceeding with the regular analysis — determine whether thereby any new equalities of lines or of angles are produced.

**169.** METHOD XI. **Geometrical propositions may be demonstrated by algebraical computation.**

At this stage of the work, the above method may be employed to establish relations between two or more angles. In case of two angles, express the value of one in algebraic form (*e.g.* $m^\circ$) and from this try to derive the value of the other.

If a relation between three or more angles has to be demonstrated, express the values of all of them excepting one by algebraic symbols (as $m^\circ$, $n^\circ$, $p^\circ$, etc.), and try to discover the value of the remaining angle. After some practice the symbols which designate the angle, as $\angle A$ or $\angle B$, may be used to express their value.

If the student is unable to find the value of the last angle when all others are given in algebraic form, he should at first attack the problem for numerical values of these angles and then try to solve the general problem.

Ex. Given $\triangle ABC$, and point $D$ in $AC$, so that $AD = AB$.

Prove $\angle DBC = \frac{1}{2}(\angle B - \angle C)$.

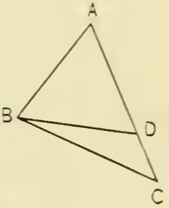
**Proof.** Let $\angle B = m^\circ$, and $\angle C = n^\circ$.

Then $\angle A = 180^\circ - m^\circ - n^\circ$.

But $\angle ABD + \angle BDA = 180^\circ - \angle A$,

*i.e.* $\angle ABD + \angle BDA = 180^\circ - (180^\circ - m^\circ - n^\circ)$

$= m^\circ + n^\circ$.



But $\angle ABD = \angle ADB$.

$\therefore 2\angle ABD = m^\circ + n^\circ$.

$\therefore \angle ABD = \frac{m^\circ}{2} + \frac{n^\circ}{2}$.

But $\angle DBC = \angle B - \angle ABD$

$$= m^\circ - \left(\frac{m^\circ}{2} + \frac{n^\circ}{2}\right)$$

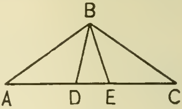
$$= \frac{m^\circ}{2} - \frac{n^\circ}{2}$$

$$= \frac{\angle B}{2} - \frac{\angle C}{2}$$

or $\angle DBC = \frac{1}{2}(\angle B - \angle C)$.

Ex. **1**. An exterior angle formed by the prolongation of the base of an isosceles triangle equals 90° increased by one half the vertex angle.

Ex. **2**. The sum of the two exterior angles of a right triangle which are formed by the prolongations of the hypotenuse equals 270°.



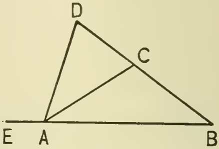
Ex. 3

Ex. **3**. If in the base $AC$ of an isosceles $\triangle ABC$, two points $D$ and $E$ be taken such that

$$AE = AB \text{ and } CD = BC,$$

prove

$$\angle DBE = \angle A.$$



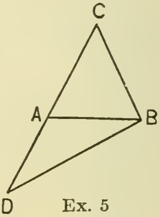
Ex. 4

Ex. **4**. If $AD = AC = CB$, and $DB$ and $EB$ are st. lines, then

$$\angle EAD = 3 \angle B.$$

Ex. **5**. If the arm $CA$ of isosceles triangle $ABC$ be produced so that $AD$ equals the base $AB$, then

$$\angle C = 180° - 4 \angle D.$$



Ex. 5

Ex. **6**. If the base $AB$ of isosceles triangle $ABC$ is produced to $D$, and $CD$ is drawn, then

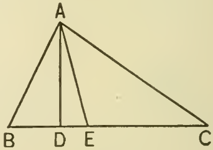
$$\angle BCD = \angle A - \angle D.$$

Ex. **7**. If in isosceles triangle $ABC$, one end $B$ of the base $AB$ is connected with a point $D$ in the opposite arm, then

$$\angle A = \tfrac{1}{2}(\angle CDB + \angle CBD).$$

Ex. **8**. If the side $AC$ of triangle $ABC$ is produced to $D$ so that $CD = CB$, then

$$\angle ABD = 90° + \tfrac{1}{2}(\angle ABC - \angle A).$$



Ex. 9

Ex. **9**. If $AD$ is the altitude and $AE$ the bisector of the angle $BAC$ of the triangle $ABC$, prove

$$\angle DAE = \tfrac{1}{2}(\angle B - \angle C).$$

Ex. **10**. The sum of three angles of a quadrilateral, diminished by the fourth exterior angle, is equal to a straight angle.

**170**. In proving an exercise, try to get as neat and as short a proof as possible.

## MISCELLANEOUS EXERCISES

Ex. **1.** What is the angle between the hands of a clock at 12 min. past 11? At 6:30? At 8:45?

Ex. **2.** The vertex angle of an isosceles triangle is 18° 44′. How many degrees are there in each base angle?

Ex. **3.** An exterior ∠ of an equiangular polygon equals one fifth of a right angle. Find the number of sides of the polygon.

Ex. **4.** Prove: *The line perpendicular to an arm of a right triangle at its middle point passes through the mid-point of the hypotenuse.*

Ex. **5.** Prove: *In any right triangle the median to the hypotenuse is one half the hypotenuse.* (Use Ex. 4.)

Ex. **6.** In $\triangle ABC$, $\angle A = 60°$, $\angle B > \angle C$. Which is the greatest side?

Ex. **7.** Construct a quadrilateral three of whose angles are 150°, 90°, and 60°.

Ex. **8.** What are some of the special properties of an isosceles triangle? Equilateral triangle? Rhombus? Rectangle? Square?

Ex. **9.** Construct $\triangle ABC$, given $AB = 3''$, $\angle A = 60°$, $\angle B = 45°$. Then construct the altitude from $A$ to $BC$, the median from $B$ to $AC$, and the bisector of $\angle C$.

Ex. **10.** What are the different tests by which to determine the congruence of triangles? The equality of angles? The equality of lines? The parallelism of lines? The perpendicularity of lines?

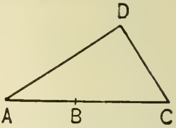
Ex. **11.** What are the different tests for determining the inequality of two lines? Of two angles?

Ex. **12.** *The bisectors of two supplementary adjacent angles are perpendicular to each other.*

Ex. **13.** If the bisectors of two adjacent angles are perpendicular to each other, the exterior sides of these angles form a straight line.

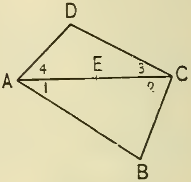
Ex. **14.** *The bisectors of vertical angles are in a straight line.*

Ex. **15.** If $\angle A + \angle C + \angle D = 180°$, then $ABC$ is a straight line.



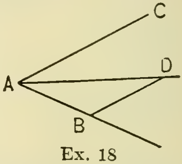
Ex. 15

Ex. **16.** If $\angle 1 + \angle 2 + \angle B = \angle 3 + \angle 4 + \angle D$, then $AEC$ is a straight line.



Ex. 16

Ex. **17.** *The altitude upon the hypotenuse of a right triangle divides the figure into two triangles which are mutually equiangular.*



Ex. 18

Ex. **18.** If through any point $D$ on the bisector of an angle $A$, a parallel be drawn to one of the sides to meet the other side in $B$, then $AB = BD$.

Ex. **19.** If from a point in the bisector of an angle, lines are drawn parallel to the sides of the angle, a rhombus is formed.

Ex. **20.** If the vertex angles of two isosceles triangles are supplementary, the base angles are complementary.

Ex. **21.** If an arm of an isosceles triangle is produced by its own length through the vertex, and the end of the prolongation is joined to the nearest end of the base, the line joining is perpendicular to the base.

Ex. **22.** *Corresponding medians of congruent triangles are equal.*

Ex. **23.** *Corresponding altitudes of congruent triangles are equal.*

Ex. **24.** *Corresponding angle-bisectors of congruent triangles are equal.*

Ex. **25.** Two isosceles triangles are congruent if the vertex angle and the altitude upon an arm of the one are respectively equal to the vertex angle and the corresponding altitude of the other.

Ex. **26.** If in the pentagon $ABCDE$,

$$AB = BC,\ AE = CD, \text{ and } \angle A = \angle C,$$

then

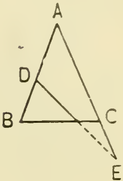
$$BE = BD, \text{ and } \angle E = \angle D.$$

Ex. **27.** *Two equilateral triangles are congruent if the altitude of one equals the altitude of the other.*

Ex. **28**. If the opposite sides of a hexagon are parallel, and two of the opposite sides are equal, all opposite sides are equal.

Ex. 28

Ex. **29**. If from the ends of the base $BC$ of an isosceles triangle $ABC$, equal parts, $BD$ and $CE$, be laid off on one arm and the prolongation of the other, the line joining $D$ and $E$ is bisected by the base.

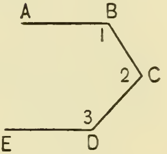


Ex. 29

Ex. **30**. If two lines are intersected by a transversal, and the bisectors of the interior angles on the same side of the transversal are perpendicular to each other, these lines are parallel.

Ex. **31**. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

Ex. **32**. If $AB \parallel ED$, $\angle 1 + \angle 2 + \angle 3 = 4$ rt. ∡.



Ex. 32

Ex. **33**. State and prove the converse of the preceding proposition.

Ex. **34**. Find the number of diagonals in a polygon of 5 sides; of 8 sides; of 10 sides; of $n$ sides.

Ex. **35**. How many sides has a polygon, the sum of whose interior angles equals three times the sum of the exterior angles? (*i.e.* one ext. $\angle$ at each vertex.)

Ex. **36**. How many sides has a polygon the sum of whose interior angles is equal to the sum of the exterior angles?

Ex. **37**. How many sides has a polygon, the sum of whose interior angles is equal to three times the sum of the angles of a hexagon?

Ex. **38**. How many sides has an equiangular polygon whose exterior angle equals the interior angle of an equilateral triangle?

Ex. **39**. If the upper base of an isosceles trapezoid is equal to one of the arms, the diagonals bisect the angles at the lower base.

Ex. **40**. If a perpendicular be dropped from the vertex to the base of a triangle, each segment of the base will be shorter than the adjacent side of the triangle.

Ex. **41**. If the vertices of a triangle lie in the sides of another triangle, the perimeter of the first is less than the perimeter of the second.

Ex. **42**. The perpendiculars from two vertices of a triangle upon the median drawn from the third vertex are equal.

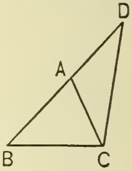
Ex. **43**. The lines joining the mid-points of the sides of a rectangle, taken in order, form a rhombus.

Ex. **44**. The lines joining the mid-points of opposite sides of any quadrilateral, taken in order, form a parallelogram.

* Ex. **45**. The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals determine the vertices of a parallelogram.

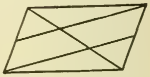
Ex. **46**. A line from the vertex of an isosceles triangle to any point in the base is shorter than either arm.

Ex. **47**. If in the triangle $ABC$, $AB > AC$, and $D$ is a point in the prolongation of $BA$, then $DB > DC$.



Ex. 47

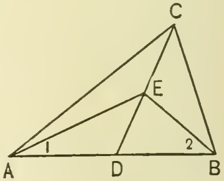
Ex. **48**. Lines joining the mid-points of two opposite sides of a parallelogram to the ends of a diagonal trisect the other diagonal.



Ex. 48

* Ex. **49**. If a point $D$ in a side $BC$ of triangle $ABC$ is joined to $A$, and $AC = BC$, $AB = AD = DC$, then $\angle C = 36°$.

* Ex. **50**. If any point $E$ in the median $CE$ is joined to $A$ and $B$ and $\angle B > \angle A$, prove that $\angle 2 > \angle 1$.



Ex. 50

Ex. **51**. *From a given point without a line, to draw a line forming with the given line an angle equal to half a right angle.*

Ex. **52**. From a given point without a line, to draw a line forming with the given line an angle of 60°. How many such lines can be drawn?

Ex. **53**. From a given point without a line to draw a line, making a given angle with the given line.

Ex. **54**. Construct a line terminating in the sides of a given angle and equal and parallel to a given line.

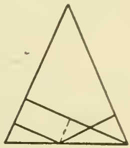
Ex. **55**. *The diagonals of an isosceles trapezoid are equal.*

* Ex. **56**. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

**171.** METHOD XII. **In order to prove that the sum of two lines, *a* and *b*, equals a third line, *c*, either**

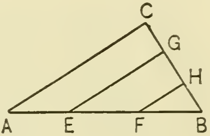
(*a*) Construct the sum of *a* and *b*, and prove the line so obtained is equal to *c*, or

(*b*) Lay off *a* (or *b*) on *c*, and prove that the line representing the difference equals *b* (or *a*).



Ex. 57

Ex. **57**. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the arms is equal to the altitude upon one of the arms.



Ex. 58

Ex. **58**. In $\triangle ABC$, if

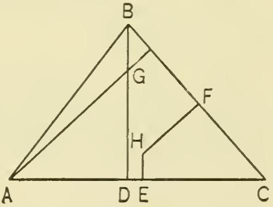
$$AE = BF$$

and $$AC \parallel EG \parallel FH,$$

prove that $$EG + FH = AC.$$

Ex. **59**. If through a point $D$ in the base $AB$ of isosceles triangle $ABC$, parallels are drawn to the arms meeting the arms in $E$ and $F$ respectively, then

$$DF + DE = AC.$$

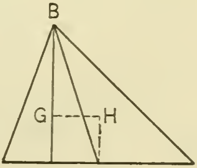


Ex. 61

Ex. **60**. The sum of the three perpendiculars dropped from any point within an equilateral triangle upon the sides is constant, and equal to the altitude of the triangle. (Ex. 57.)

* Ex. **61**. If the altitude $BD$ of $\triangle ABC$ is intersected by another altitude in $G$, and $EH$ and $HF$ are perpendicular-bisectors, prove

$BG = 2(HE)$, and $AG = 2(HF)$.



Ex. 62

Ex. **62**. The line joining the point of intersection of the altitudes of a triangle and the point of intersection of the three perpendicular-bisectors cuts off one third of the corresponding median. (Ex. 61.)

* Ex. **63**. The points of intersection of the altitudes, medians, and perpendicular-bisectors of a triangle lie in a straight line.

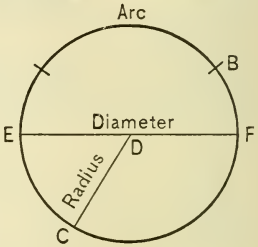
# BOOK II

## THE CIRCLE — CONSTRUCTIONS

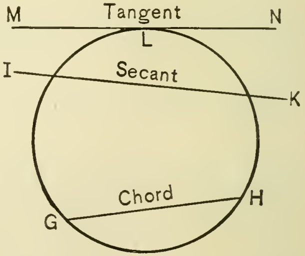
**172.** Def. A **circle** is a plane closed curve, having all its points equally distant from a fixed point in the plane, as *ABC*(37).

**173.** Def. The fixed point (*D*) is the **center**. The length of the circle is called the **circumference**. An **arc** is any portion of the circle, as *AB*. A **semicircle** is half of the circle. A **minor arc** is an arc less than a semicircle; a **major arc** is an arc greater than a semicircle.

The word *arc* taken alone generally signifies a minor arc.



**174.** Def. A **radius** is a line joining the center to any point in the circle, as *DC*. A **diameter** is a straight line through the center terminated at each end by the circle, as *EF*.



**175.** Def. A **chord** is a straight line joining any two points in the circle, as *GH*. A **secant** is a straight line that *intersects* the circle in two points, as *IK*. A **tangent** is a straight line that *touches* the circle at one point only, and does not intersect it if produced, as *MN*. This point (*L*) is called the *point of contact* or *point of tangency*.

**176**. Def. A **central angle** is an angle formed by two radii, as $\angle EDC$.

An angle is said to *intercept* any arc that is cut off by its sides, and this arc is said to *subtend* the angle. A chord that joins the ends of an arc is said to *subtend* this arc; thus chord $GH$ subtends the arc $\overset{\frown}{GH}$.

**177**. Def. Circles having the *same center* are called **concentric** circles.

## PRELIMINARY THEOREMS

**178**. *All radii of the same circle are equal.* (By definition.)

**179**. *A point is within a circle if its distance from the center is less than a radius. A point is on (or in) the circle if its distance from the center is equal to a radius. A point is without a circle if its distance from the center is greater than a radius.*

**180**. *Two circles are equal if their radii are equal.* (Prove by superposition.)

**181**. *All diameters of the same or of equal circles are equal.*

**182**. *A diameter bisects the circle.* (Prove by superposition.)

Note: 1. Diameters are *central* chords. All other chords are *non-central* chords.

2. A circle is determined by its center and a radius.

---

Ex. **1**. Construct a circle with center $A$ and radius equal to $r$.

Ex. **2**. Construct a central angle of 90°, 60°, 45°, 30°.

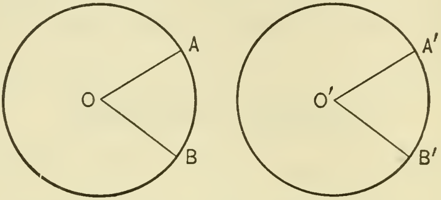
Ex. **3**. Prove: Every diameter bisects the circle.

Ex. **4**. Prove: A circle cannot have two centers.

Ex. **5**. Construct two concentric circles, then construct a circle concentric to the first two circles and midway between them.

### PROPOSITION I. THEOREM

**183.** *In the same circle or in equal circles, equal central angles have equal arcs; and, conversely, equal arcs have equal central angles.*



I. **Given** in equal Ⓢ, $O$ and $O'$, $\angle O = \angle O'$.

**To prove** $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Place circle $O$ on its equal circle $O'$ so that $OB$ coincides with $O'B'$. | 1. By hyp., the Ⓢ are =.<br>Axiom 19. |
| 2. $OA$ falls along $O'A'$. | 2. By hyp., $\angle O = \angle O'$. |
| 3. $A$ coincides with $A'$. | 3. Radii of = Ⓢ are =. |
| 4. $\overset{\frown}{AB}$ coincides with $\overset{\frown}{A'B'}$. | 4. Radii of = Ⓢ are =. |
| 5. $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$. | 5. (19). |

II. CONVERSELY. **Given** in equal Ⓢ, $O$ and $O'$, $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

**To prove** $\angle O = \angle O'$.

| | |
|---|---|
| 1. Statement (1) above. | 1. By hyp., the Ⓢ are =. |
| 2. $\overset{\frown}{BA}$ falls along $\overset{\frown}{B'A'}$. | 2. Equal Ⓢ have = radii. |
| 3. Point $A$ falls on $A'$. | 3. By hyp., $\overset{\frown}{AB} = \overset{\frown}{A'B'}$. |
| 4. $OA$ coincides with $O'A'$. | 4. Two pts. det. a line. |
| 5. $\therefore \angle O = \angle O'$. | 5. (19). |

**184.** COR. *In the same or in equal circles, the greater of two unequal central angles intercepts the greater arc, and conversely.* HINT. By superposition.

[Practical applications, pages 287–290.]

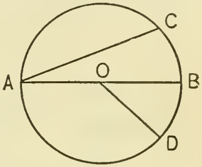
**185.** METHOD XIII. **The equality of arcs in the same circle or equal circles can be demonstrated by the equality of their central angles.**

---

Ex. **1.** If $AB$ and $CD$ are diameters of the same circle, arc $AC$ = arc $BD$.

Ex. **2.** If $AB$ is a diameter, $OD$ a radius, $AC$ a chord, and $\angle BOD = 2\angle A$, then $\widehat{BD} = \widehat{BC}$.

Ex. **3.** If a diameter $AB$ bisects the angle $A$, which is formed by two chords $AC$ and $AD$, then $\widehat{BC} = \widehat{BD}$.



Ex. 2

Ex. **4.** If from a point $A$ in a circle a chord $AB$ and a diameter $AC$ are drawn, a radius parallel to $AB$ bisects $\widehat{BC}$.

Ex. **5.** If through a point equidistant from two points in a circle a radius is drawn, the arc between the two points is bisected.

Ex. **6.** If the perpendiculars drawn from a point in the circle upon two radii are equal, the point bisects the arc intercepted by the two radii.

Ex. **7.** Divide a circle into four equal parts; into 8; into 6.

**186.** A circle may be divided into 360 equal parts called *degrees*. A degree may be subdivided into 60 equal parts called *minutes*, and similarly a minute may be divided into 60 *seconds*.

An angle is *measured by* an arc, if both angle and arc have the same number of degrees.

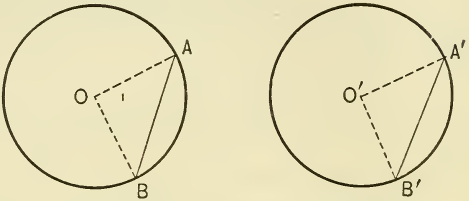
---

Ex. **8.** Prove that a central angle is measured by its intercepted arc if the central angle equals: (*a*) 90°; (*b*) 30°; (*c*) 1°.

Ex. **9.** Construct an arc of (*a*) 45°, (*b*) 60°, (*c*) 150°.

### PROPOSITION II. THEOREM

**187.** *In the same circle, or in equal circles, equal arcs have equal chords; and, conversely, equal chords have equal arcs.*



I. **Given** in equal ⓢ, $O$ and $O'$, $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

**To prove** $AB = A'B'$.

**Proof**

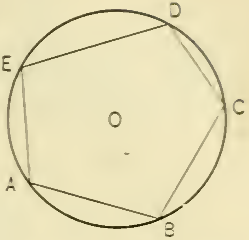
| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $OA, OB, O'A', O'B'$. | **1.** Ax. 13. |
| **2.** $\overset{\frown}{AB} = \overset{\frown}{A'B'}$ | **2.** Hyp. |
| **3.** $\angle AOB = \angle A'O'B'$. | **3.** In equal ⓢ, equal arcs have equal central ∡. |
| **4.** $AO = A'O'$, $BO = B'O'$. | **4.** Radii of equal ⓢ are =. |
| **5.** $\triangle ABO \cong \triangle A'B'O'$. | **5.** s. a. s. = s. a. s. |
| **6.** $\therefore AB = A'B'$. | **6.** Corr. sides of congruent △ are =. |

II. CONVERSELY. **Given** in equal ⓢ, $O$ and $O'$, $AB = A'B'$.

**To prove** $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

| | |
|---|---|
| **1.** Draw $OA, OB, O'A', O'B'$. | **1.** A st. line etc. |
| **2.** $AB = A'B'$. | **2.** Hyp. |
| **3.** $AO = A'O'$, $BO = B'O'$. | **3.** Radii of = ⓢ are =. |
| **4.** $\triangle ABO \cong \triangle A'B'O'$. | **4.** s. s. s. = s. s. s. |
| **5.** $\angle AOB = \angle A'O'B'$. | **5.** Corr. ∡ of ≅ △ are =. |
| **6.** $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$. | **6.** In = ⓢ, equal central ∡ have = arcs. |

**188.** Def. A polygon is **inscribed in a circle**, if all its vertices are in the circle, as *ABCDE*. The circle is then said to be **circumscribed about the polygon.**



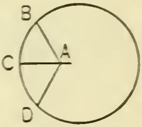
The center of the circumscribed circle is called the **circumcenter** of the polygon.

**189.** Method XIV. **The equality of arcs, in the same circle or equal circles, may also be demonstrated by the equality of their chords, and vice versa.**

---

Ex. **1.** If $\angle BAC = \angle DAC$ and $AB = AD$, then $\overset{\frown}{BC} = \overset{\frown}{CD}$.

Ex. **2.** If $\angle BAC = \angle DAC$ and $\angle CBA = \angle CDA$ (not drawn in the figure), then $\overset{\frown}{CB} = \overset{\frown}{CD}$.



Exs. 1, 2

Ex. **3.** If $\triangle ABC$ is inscribed in a circle and $\angle A = \angle B$, prove that $\overset{\frown}{AC} = \overset{\frown}{BC}$.

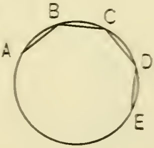
Ex. **4.** If two chords bisect each other, the arcs intercepted by the sides of a pair of vertical angles are equal.

Ex. **5.** *If two chords bisect each other, they are diameters*

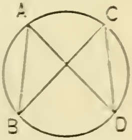
Ex. **6.** If chords $AB$, $BC$, $CD$, $DE$ are equal, then chords $AC$, $BD$, $CE$ are equal.

Ex. **7.** If $AB = CD$, then $BC = AD$.

Ex. **8.** If two intersecting chords, $AD$ and $BC$, are equal, then $AB = CD$.



Ex. 6



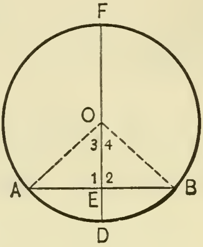
Exs. 7, 8

Ex. **9.** The diagonals of an equilateral pentagon inscribed in a circle are equal.

Ex. **10.** The radii drawn to the vertices of an inscribed equilateral hexagon divide the figure into six congruent equilateral triangles.

## Proposition III. Theorem

**190.** *The diameter perpendicular to a chord bisects the chord and the arcs of the chord.*



**Given** in ⊙ $O$, the diam. $FD \perp AB$.

**To prove** $AE = EB$, $\overset{\frown}{AD} = \overset{\frown}{DB}$, $\overset{\frown}{AF} = \overset{\frown}{FB}$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $OA$, $OB$. | **1.** Ax. 13. |
| **2.** $OE \perp AB$. | **2.** Hyp. |
| **3.** $\angle 1 = \angle 2 =$ rt. $\angle$. | **3.** $\perp$s form right $\angle$s. |
| **4.** $OA = OB$. | **4.** Radii of a ⊙ are =. |
| **5.** $OE = OE$. | **5.** Iden. |
| **6.** Rt. $\triangle AOE \cong$ rt. $\triangle BOE$. | **6.** hy. a. = hy. a. |
| **7.** $\therefore AE = EB$, $\angle 3 = \angle 4$. | **7.** Corr. pts. of ≅ △s are =. |
| **8.** $\therefore \overset{\frown}{AD} = \overset{\frown}{DB}$. | **8.** Equal cent. $\angle$s have = arcs. |
| **9.** $\overset{\frown}{DAF} = \overset{\frown}{DBF}$. | **9.** A diameter bisects the ⊙. |
| **10.** $\therefore \overset{\frown}{AF} = \overset{\frown}{FB}$. | **10.** Equals from equals, rem. =. |

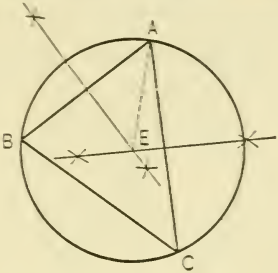
**191.** Cor. *The perpendicular-bisector of a chord passes through the center of the circle.*

---

Ex. If in circle $O$, $AB = AC$, $OD \perp AB$, and $OE \perp AC$, prove that $\triangle ADO \cong \triangle AEO$.

### PROPOSITION IV. PROBLEM

**192.** *To circumscribe a circle about a given triangle.*



**Given** △*ABC*.

**Required** to circumscribe a circle about △*ABC*.

**Construction.** Draw the perpendicular-bisectors of the sides *AC* and *AB*. They will intersect at some point *E*. (82) (102)

From *E* as a center, with a radius equal to *EA*, describe a circle *ABC*.

*ABC* is the required circle.

**Proof.** *E* is equidistant from *A*, *B*, and *C*. (162)

∴ A circle drawn from *E* as a center with a radius equal to *EA* will pass through *A*, *B*, and *C*.

**193.** COR. 1. *Three points not in a straight line determine a circle.*

**194.** COR. 2. *A circle cannot be drawn through three points which lie in the same straight line.*

**195.** COR. 3. *A straight line cannot intersect a circle in more than two points.*

**196.** COR. 4. *Two circles cannot meet in more than two points.*

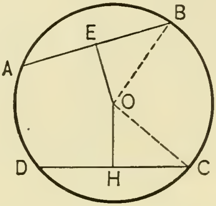
---

EX. **1.** *To find the center of a given circle.*

EX. **2.** *To bisect a given arc.*

## PROPOSITION V. THEOREM

**197.** *In the same circle, or in equal circles, equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.*



I. **Given** in $\odot ABCD$, $AB = CD$, $OE \perp AB$, $OH \perp CD$.

**To prove** $OE = OH$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $OB$ and $OC$. | 1. A st. line may be drawn between two pts. |
| 2. $AB = CD$, $OE \perp AB$, $OH \perp DC$. | 2. Hyp. |
| 3. $E$ and $H$ are mid-pts. of the chords. | 3. Dia. $\perp$ chord bisects the chord. |
| 4. $EB = HC$. | 4. Halves of equals are = |
| 5. $OB = OC$. | 5. Radii of a $\odot$ are =. |
| 6. $\angle BEO = \angle OHC =$ a rt. $\angle$. | 6. $\perp$s form rt. $\angle$s. |
| 7. Rt. $\triangle OEB \cong$ rt. $\triangle OHC$. | 7. hy. a = hy. a. |
| 8. $\therefore OE = OH$. | 8. Corr. sides of $\cong$ $\triangle$s are = |

---

Ex. 1. The perpendiculars from the center to the sides of an inscribed equilateral polygon are equal.

Ex. 2. Prove proposition V by drawing $AO$ and $DO$ and then by proving $\triangle AOB \cong \triangle DOC$.

HINT. See p. 98, Ex. 23.

II. CONVERSELY. Given in ⊙ $ABCD$,
$OE = OH, OE \perp AB, OH \perp CD.$

To prove $AB = CD.$

| | |
|---|---|
| 1. Draw $OB$ and $OC$. | 1. Ax. 13. |
| 2. $OB = OC$. | 2. Radii same ⊙ are =. |
| 3. $OE = OH$, $OE \perp AB$, $OH \perp CD$. | 3. Hyp. |
| 4. $\angle BEO = \angle OHC =$ rt. $\angle$ | 4. ⊥s form rt. ∠s. |
| 5. Rt. $\triangle OEB \cong$ rt. $\triangle OHC$. | 5. hy. a. = hy. a. |
| 6. $EB = HC$. | 6. Corr. sides ≅ △s are =. |
| 7. But $E$ and $H$ are mid-pts. of the chords. | 7. Prop. III. |
| 8. $\therefore AB = CD$. | 8. Why? |

**198.** METHOD XV. **The equality of two chords in the same circle, or in equal circles, is usually established by means of equal distances from the center or equal subtended arcs.**

---

Ex. **1.** If through any point in a radius two chords are drawn making equal angles with the radius, these chords are equal.

Ex. **2.** A line joining the point of intersection of two equal chords to the center bisects the angle formed by the chords.

Ex. **3.** In a given circle, to draw a chord equal and parallel to a given chord.

Ex. **4.** In a given circle, to draw a chord equal to a given chord, and parallel to a given line.

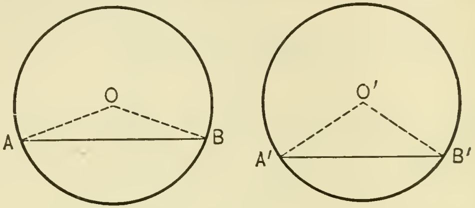
Ex. **5.** At the ends of a diameter, chords are drawn making equal angles with the diameter. Prove they are equal.

Ex. **6.** A diameter bisects each of two non-central chords. Show the chords are parallel.

Ex. **7.** Two intersecting chords have a segment of each equal. Prove the chords equal.

## PROPOSITION VI. THEOREM

**199.** *In the same circle, or in equal circles, the greater of two minor arcs has the greater chord; and conversely, the greater chord has the greater arc.*



I. **Given** in equal Ⓢ, $O$ and $O'$,

$$\overset{\frown}{AB} > \overset{\frown}{A'B'}.$$

**To prove** $AB > A'B'$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $OA$, $OB$, $O'A'$, $O'B'$. | 1. Why? |
| 2. $\overset{\frown}{AB} > \overset{\frown}{A'B'}$. | 2. Hyp. |
| 3. In $\triangle AOB$ and $A'O'B'$, $\angle O > \angle O'$. | 3. (184). |
| 4. $AO = A'O'$ and $BO = B'O'$. | 4. Why? |
| 5. $\therefore AB > A'B'$. | 5. (132). |

II. CONVERSELY **Given** in equal Ⓢ, $O$ and $O'$,

$$AB > A'B'.$$

**To prove** $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

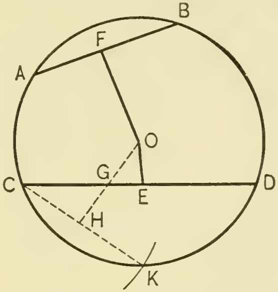
| | |
|---|---|
| 1. Draw $OA$, $OB$, $O'A'$, $O'B'$. | 1. Why? |
| 2. $AB > A'B'$. | 2. Hyp. |
| 3. $OA = O'A'$ and $OB = O'B'$. | 3. Why? |
| 4. In $\triangle AOB$ and $A'O'B'$, $\angle O > \angle O'$. | 4. (133). |
| 5. $\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}$. | 5. (184). |

NOTE. Not all proofs will be complete hereafter. The student must supply the parts wanting.

[See p. 115 for exercises.]

### PROPOSITION VII. THEOREM

**200.** *In the same circle, or in equal circles, if two chords are unequal, the greater chord is at the less distance from the center.*



**Given** in ⊙ $ABDC$, $CD > AB$, $OE \perp CD$, and $OF \perp AB$.

**To prove** $OE < OF$.

**Proof**

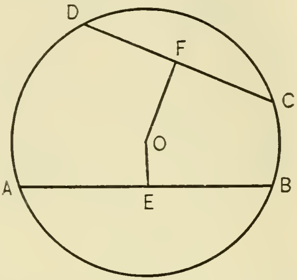
| STATEMENTS | REASONS |
|---|---|
| 1. $CD > AB$. | 1. Hyp. |
| 2. $\overset{\frown}{CD} > \overset{\frown}{AB}$. | 2. (199). |
| 3. From $C$ as center, and radius equal to $AB$, strike off an arc to cut $\overset{\frown}{CD}$ within at $K$. | 3. Why? |
| 4. Draw $CK$, and $OH \perp CK$. | 4. Why? |
| 5. $CK = AB$. | 5. Why? |
| 6. $OH = OF$. | 6. Equal chords are = dist. from center. |
| 7. $OE < OG$. | 7. (131). |
| 8. $OG < OH$. | 8. (Ax. 11). |
| 9. $\therefore OE < OH$ | 9. (Ax. 9). |
| or $OE < OF$. | |

[See p. 115 for exercises.]

### Proposition VIII. Theorem

**201.** *In the same circle, or in equal circles, if two chords are unequally distant from the center, the chord at the less distance is the greater.*

[Converse to Prop. VII.]



**Given** in ⊙ $ABCD$ with $OE < OF$, $OE \perp AB$, $OF \perp DC$.

**To prove** $AB > DC$.

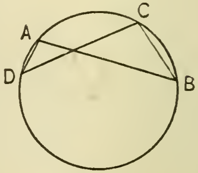
**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Either $AB < DC$, or $AB = DC$, or $AB > DC$. | **1.** Why? |
| **2.** If $AB < DC$, then $OE > OF$. | **2.** (200). |
| **3.** If $AB = DC$, then $OE = OF$. | **3.** (197). |
| **4.** Statements 2, 3 are false. | **4.** By hyp., $OE < OF$. |
| **5.** $\therefore AB > DC$. | **5.** Why? |

---

Ex. **1.** If $AB > CD$, and $\overset{\frown}{ACB}$ and $\overset{\frown}{DAC}$ are minor arcs, prove that $CB > AD$.

Ex. **2.** If $CB > AD$, and $\overset{\frown}{DAC}$ and $\overset{\frown}{ACB}$ are minor arcs, prove that $AB > CD$.



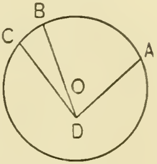
Exs. 1, 2

**202.** Method XVI. **The inequality of chords, in the same circle or in equal circles, is usually established by means of unequal distances from the center or by means of unequal arcs.**

---

Ex. **1.** If $ABC$ is any triangle inscribed in a circle and $\angle A > \angle B$, then $\overset{\frown}{BC} > \overset{\frown}{AC}$.

Ex. **2.** State and prove the converse of Exercise 1.



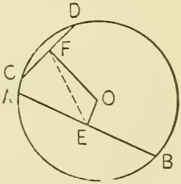
Exs. 3, 4

Ex. **3.** If $AD = DC$,
and $\angle ADB > \angle BDC$,
then $\overset{\frown}{AB} > \overset{\frown}{BC}$.

Ex. **4.** If $AD = DC$, and $\overset{\frown}{AB} > \overset{\frown}{BC}$, then $\angle ADB > \angle BDC$.

Ex. **5.** If upon a radius two perpendicular chords are drawn, the one nearer the center is the greater chord.

Ex. **6.** The perpendicular from the center of a circle to a side of an inscribed equilateral hexagon is less than the perpendicular from the center to the side of an inscribed equilateral octagon.



Ex. 7

Ex. **7.** If $EO \perp AB$, $OF \perp CD$, and $\angle OEF > \angle OFE$, then $AB > CD$.

Ex. **8.** If $\triangle ABC$ is inscribed in a circle and $\angle B > \angle C$, then the perpendicular from the center upon $AB$ is greater than the perpendicular from the center upon $AC$.

Ex. **9.** The shortest chord through a point within a circle is the chord perpendicular to the radius drawn through the point.

Ex. **10.** Two chords drawn from a point in a circle are unequal if they make unequal angles with the radius drawn to that point.

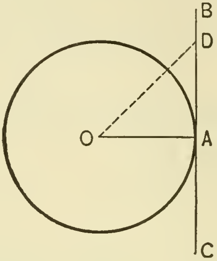
* Ex. **11.** Two chords drawn through an interior point are unequal if they make unequal angles with the radius drawn through that point

Ex. **12.** *A diameter is greater than any noncentral chord.*

Hint. Draw radii to ends of chord. Their sum equals a diameter.

## PROPOSITION IX. THEOREM

**203.** *The straight line perpendicular to a radius at its outer extremity is a tangent to the circle.*



**Given** in $\odot O$, radius $OA \perp BC$ at $A$.

**To prove** $BC$ is a tangent.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Join any other point $D$ in $BC$ to $O$. | 1. Why? |
| 2. $OD > OA$. | 2. (131). |
| 3. $D$ lies without the circle. | 3. (179). |
| 4. $\therefore BC$ is a tangent to $\odot O$. | 4. (175). |

**204.** COR. 1. *A tangent is perpendicular to the radius drawn to the point of contact.* (Converse to Prop. IX.)

HINT. The shortest line is the perpendicular, since the $\perp$ is the shortest line (identity).

**205.** COR. 2. *The perpendicular to a tangent at the point of contact passes through the center of the circle.*

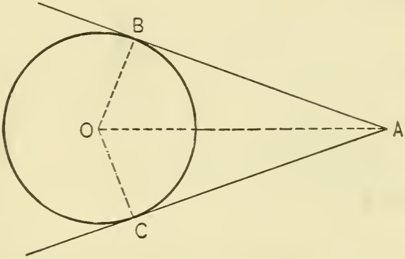
**206.** COR. 3. *The perpendicular from the center to a tangent meets it at the point of contact.*

**207.** COR. 4. *At a given point of contact there can be one tangent only.*

**208.** Def. The **length of a tangent** drawn from a point to a circle is the length of the line from this point to the point of contact, as $AB$ (Prop. X).

### Proposition X. Theorem

**209.** *The tangents drawn to a circle from a point without are equal.*



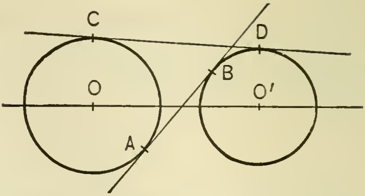
**Given** tangents $AB$ and $AC$ to circle $O$.

**To prove** $AB = AC$.

### Proof

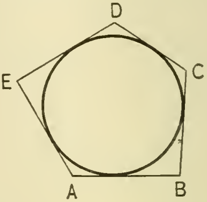
| STATEMENTS | REASONS |
|---|---|
| 1. Draw $OA$, $OB$, $OC$. | 1. Why? |
| 2. $\angle OBA$ and $\angle OCA$ are rt. $\angle$. | 2. Prop. IX, Cor. 1. |
| 3. In rt. $\triangle OBA$ and $OCA$, $OB = OC$. | 3. Why? |
| 4. $OA = OA$. | 4. Why? |
| 5. $\triangle OBA \cong \triangle OCA$. | 5. hy. a. = hy. a. |
| 6. $\therefore AB = AC$. | 6. Why? |

**210.** Cor. *The tangents drawn to a circle from a point without make equal angles with the line joining the point to the center* (*i.e.* $\angle OAB = \angle OAC$).



**211.** Def. A common tangent to two circles is called an **internal tangent** if the centers ($O$ and $O'$) are on opposite sides of the tangent, as $AB$. Otherwise it is called a **common external tangent**, as $CD$.

The **length** of a common tangent is the length of the segment between the **points of contact.**



**212.** Def. A polygon is **circumscribed about a circle** if all its sides are tangent to the circle, as $ABCDE$. The **circle** is then said to be **inscribed** in the polygon.

The center of the inscribed circle is called the *incenter* of the polygon.

---

Ex. **1.** *The common internal tangents of two circles are equal.*

Ex. **2.** *The common external tangents of two circles are equal.*

Ex. **3.** A chord forms equal angles with the tangents drawn at its ends.

Ex. **4.** The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Ex. **5.** If hexagon $ABCDEF$ is circumscribed about a circle, $AB + CD + EF = BC + DE + FA$.

Ex. **6.** The sum of the arms of a right triangle circumscribed about a circle is equal to the hypotenuse increased by the diameter of the circle.

Ex. **7.** If two tangents make an angle of 60°, the chord joining the points of contact equals either tangent.

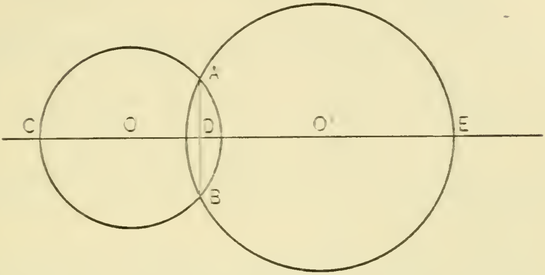
* Ex. **8.** Triangle $ABC$ is circumscribed about a circle, which touches the sides $AB$, $BC$, and $CA$ in $X$, $Y$, and $Z$, respectively. Find the lengths of $AX$, $BY$, and $CZ$ if $AB = 3$, $BC = 4$, and $CA = 5$.

**213.** Def. The **line of centers** is the line determined by the centers of two circles.

Thus $OO'$ is the line of centers of circles $O$ and $O'$. See figure at top of page.

### PROPOSITION XI. THEOREM

**214.** *If two circles intersect, the line of centers bisects their common chord at right angles.*



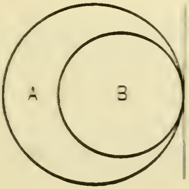
**Given** circles $ACB$ and $AEB$ intersecting at $A$ and $B$.

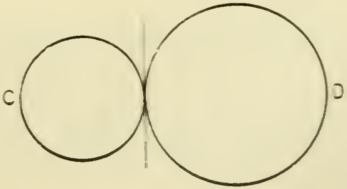
**To prove** $OO'$, their line of centers, is the perpendicular-bisector of $AB$.

**Proof.** $O$ and $O'$ are each equally distant from $A$ and $B$. (Why?)

∴ $OO'$ is the perpendicular-bisector of $AB$. (79)

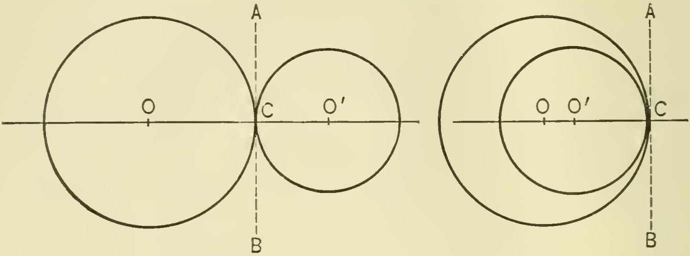
**215.** DEF. Two circles are **tangent to each other** if both are tangent to the same straight line at the same point. They are tangent internally or externally, according as one circle lies within or without the other. Thus, circles $A$ and $B$ are tangent internally, and circles $C$ and $D$ are tangent externally.





### Proposition XII. Theorem

**216.** *If two circles are tangent to each other, their line of centers passes through the point of contact.*



**Given** $OO'$ is the line of centers of circles $O$ and $O'$ that are tangent at $C$.

**To prove** $OO'$ passes through $C$.

**Proof.** Draw common tangent $AB$ at $C$. At the point of contact, $C$, draw a perpendicular to the common tangent.

This perpendicular passes through $O$ and $O'$. (205)

$\therefore$ $OO'$ passes through $C$. (Why?)

---

Ex. **1.** What are the relative positions of two circles, if the distance between their centers is

(*a*) Greater than the sum of the radii?
(*b*) Equal to the sum of the radii?
(*c*) Less than the sum but greater than the difference of the radii?
(*d*) Equal to the difference of the radii?
(*e*) Less than the difference of the radii?
(*f*) Equal to zero?

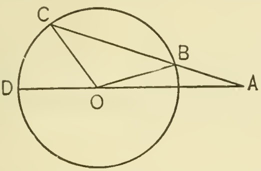
Ex. **2.** If in the diagram of the preceding proposition (XII) tangents are drawn from $A$ to circles $O$ and $O'$, these tangents are equal.

Ex. **3.** If a secant intersects two concentric circles, its segments, intercepted by the two circles, are equal.

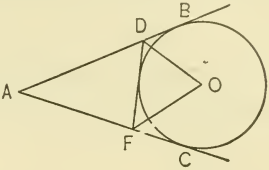
Ex. **4.** Two parallel chords, drawn from the extremities of a diameter, are equal.

Ex. **5**. If the radius $OB$ is equal to $AB$, prove $\angle COD = 3\angle A$.

Ex. **6**. If from $A$ the tangents $AB$ and $AC$ are drawn to a given circle $O$, and a third tangent intersects $AB$ and $AC$ in $D$ and $F$,



Ex. 5

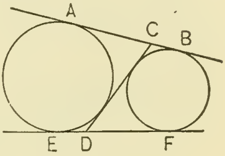


Ex. 6

**Prove** $AD + DF + AF = 2\,AB$,

and $\angle O = \text{rt. } \angle - \dfrac{\angle A}{2}$.

* Ex. **7**. The two common external tangents of two circles intercept on a common internal tangent a segment, $CD$, equal to the external tangent, $AB$.



Ex. 7

## MEASUREMENT

**217**. To **measure** a quantity is to find how many times it contains another quantity of the same kind, called a **unit of measure.**

**218**. Def. The **numerical measure** of a quantity is the number that expresses how many times the quantity contains the unit.

Thus a line is measured by finding the number of yards or inches that it contains. If the result is 16 inches, then 16 is the numerical measure of the line.

**219**. Def. A number is **rational** if it can be expressed as a common fraction, proper or improper.

Obviously this definition includes all integers.

Thus, $\frac{2}{7}$, 9, $2\frac{3}{4}$, 11.4 are rational numbers. Any rational number can be represented in the form $\frac{m}{n}$, in which $m$ and $n$ are integers.

**220.** Def. A number is **irrational** if it cannot be expressed by a common fraction, proper or improper.[1]

Thus $\sqrt{2}$ cannot be a fraction, for if $\sqrt{2}$ were equal to $\frac{m}{n}$ (where $m$ and $n$ are integers that have no common factor), then $2 = \frac{m \cdot m}{n \cdot n}$, a condition which is impossible since $m$ and $n$ have no common factors. Hence $\sqrt{2}$ is an irrational number. Other irrational numbers are $\frac{\sqrt{7}}{5}$, $\sqrt[3]{9}$, $\sqrt[5]{11}$, etc.

**221.** In the following chapters we shall designate as *successive approximate values* the decimals that we obtain if we calculate the values of irrational numbers to one, two, three, etc., decimal places.

Thus the successive approximate values of $\sqrt{2}$ are:

$$1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad \text{etc.}$$

Since the true value of $\sqrt{2}$ lies between 1.4 and 1.5, the first approximation differs from the true value by less than .1. Similarly the second approximation differs from the true value by less than .01, the third by less than .001, etc. It is evident that by continuing the decimal this difference may be diminished to less than one millionth or one billionth or less than any assigned value, however small.

**222.** Irrational numbers cannot be expressed by a common fraction, proper or improper, but approximate values can be found that differ from the true value by less than any assigned value, however small.

**223.** Def. Two irrational numbers are called *equal* if *all* their successive approximate values are respectively equal.

Obviously two numbers that satisfy this condition cannot differ by any value, however small.

---

[1] The definition of "irrational" as given in many texts on elementary algebra refers to a special kind of irrational numbers only, viz. those that can be obtained by evolution. This is the only kind of irrational numbers that can be obtained by the operations of elementary algebra.

Thus, it can be shown that all successive approximate values of $\sqrt{3}$ and $\frac{3}{\sqrt{3}}$ are respectively equal, no matter to how many places of decimals we carry the calculation. Hence we call $\sqrt{3}$ and $\frac{3}{\sqrt{3}}$ equal irrational numbers.

**224.** DEF. The **ratio** of two quantities of the same kind is equal to the quotient of their numerical measures expressed in terms of the same unit.

Thus, the ratio of two quantities, $a$ and $b$, is $\frac{a}{b}$ or $a \div b$; the ratio of four yards and two yards is $\frac{4}{2}$, or 2. A ratio is used to compare the magnitude of two quantities.

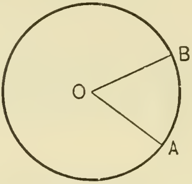
**225.** If the ratio of two quantities is a rational number, there always exists a third quantity, called a *common measure*, which is contained an integral number of times in each.

Thus, if $AB$ and $CD$ designate the length of two lines expressed in terms of the same unit, and $\frac{AB}{CD} = \frac{5}{9}$, then the ninth part of $CD$ is contained in $AB$ five times. Or if $CD$ be divided into nine equal parts and one of these parts be laid off on $AB$, as often as possible, then $AB$ contains 5 of these parts. Each of these parts is a common measure of $AB$ and $CD$.

**226.** If the ratio of two quantities is an irrational number, it can be shown that they have no common measure; *e.g.* if $\frac{AB}{CD} = \sqrt{7}$, $AB$ and $CD$ cannot have a common rational measure, for otherwise $\sqrt{7}$ would equal a rational number.

### Proposition XIII. Theorem[1]

**227.** *A central angle is measured by its intercepted arc.*



**Given** central $\angle O$ intercepting $\widehat{AB}$.

**To prove** $\angle O$ and $\widehat{AB}$ are measured by the same number of degrees.

**Proof.** Case I. The number of degrees in $\angle O$ is rational.

Then the number of degrees in $\angle O$ can be represented by $\frac{m}{n}$, where $m$ and $n$ are integers.

If we construct 360 equal angles about $O$, each angle would intercept $\frac{1}{360}$ of the circle. (183)

Or, A central $\angle$ of 1° intercepts an arc of 1°.

If the angle of 1° be divided into $n$ equal parts, all intercepted arcs would be equal.

Or, A central $\angle$ of $\left(\frac{1}{n}\right)^\circ$ intercepts an arc of $\left(\frac{1}{n}\right)^\circ$.

Taking $m$ of these parts, we have

$$\text{A central } \angle \text{ of } \left(\frac{m}{n}\right)^\circ \text{ intercepts an arc of } \left(\frac{m}{n}\right)^\circ.$$

Case II. The numbers of degrees in $\angle O$ is irrational.

Let the number of degrees in $\angle O$ be some irrational number, *e.g.* $\sqrt{2} = 1.41421 \cdots$, and let us consider the successive approximate values of this number.

---

[1] Teachers who prefer the proof which is based upon the principle of limits will find the same in the appendix, pp. 275–277.

If $\angle O$ contained 1.4°, $\overset{\frown}{AB}$ contains 1.4°. (Case I.)

If $\angle O$ contained 1.41°, $\overset{\frown}{AB}$ contains 1.41°. (Case I.)

If $\angle O$ contained 1.414°, $\overset{\frown}{AB}$ contains 1.414°. (Case I.)

In other words all approximations of the numerical measures of $\angle O$ and of $\overset{\frown}{AB}$ are respectively equal.

$\therefore$ the numerical measures of $\angle O$ and $\overset{\frown}{AB}$ are equal. (223)

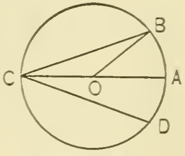
Note. In the above demonstration a concrete numerical value is assumed for $\angle O$. It is evident, however, that the proof does not at all depend upon the particular value selected; in other words, that the conclusion follows in like manner for any other numerical value of $\angle O$. Hence the proof is general. For a more general form of proofs of this type see § 347.

**228.** Cor. *In the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs.*

---

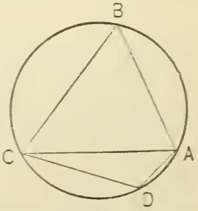
Ex. **1.** If $\overset{\frown}{AB} = 40°$, how many degrees are (*a*) in central $\angle AOB$? (*b*) in $\angle ACB$?

Ex. **2.** If $\overset{\frown}{BD} = 60°$, how many degrees are in $\angle C$?



Exs. 1, 2

**229.** Def. An **inscribed angle** of a circle is an angle whose vertex lies in the circle and whose sides are chords; as $\angle B$, $\angle D$, $\angle BCD$, $\angle BCA$.

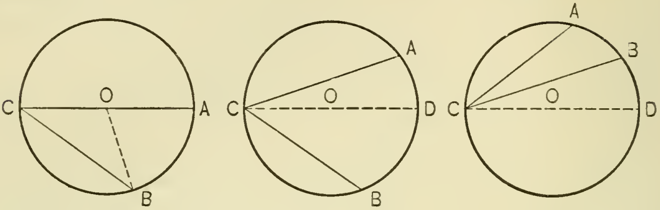


**230.** Def. A **segment** of a circle is a figure formed by an arc and its chord, as arc *CBA* and chord *CA*.

**231.** Def. An *angle* is said to be **inscribed in a segment** if its vertex lies in the arc and its sides pass through the extremities of that arc. Thus $\angle D$ is inscribed in segment *CDA*, and $\angle B$ in segment *CBA*.

### PROPOSITION XIV. THEOREM

**232.** *An inscribed angle is measured by one half the intercepted arc.*



**Given** $\angle ACB$ inscribed in $\odot O$.

**To prove** $\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.

CASE I. One side of the angle is a diameter of the circle.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $OB$. | **1.** Why? |
| **2.** $BO = OC$. | **2.** Why? |
| **3.** $\angle C = \angle B$. | **3.** Base $\measuredangle$ of an isos. $\triangle$ are $=$. |
| **4.** $\angle AOB = \angle C + \angle B$. | **4.** (120). |
| **5.** $\angle AOB = 2\angle C$. | **5.** Sub. |
| **6.** $\angle ACB = \frac{1}{2}\angle AOB$. | **6.** Why? |
| **7.** $\angle AOB$ is measured by $\overset{\frown}{AB}$. | **7.** A central $\angle$ is meas. by its arc. |
| **8.** $\therefore$ $\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$. | **8.** Why? |

CASE II. The center lies within the angle.

Draw the diameter $CD$.

Then $\angle ACD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$. (Case I.)

and $\angle BCD$ is measured by $\frac{1}{2}\overset{\frown}{BD}$. (Case I.)

$\therefore \angle ACD + \angle BCD$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BD})$. (Ax. 2.)

Or $\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.

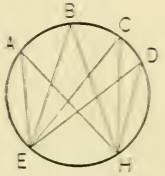
CASE III. The center lies without the angle.

Draw the diameter $CD$.

Then $\angle ACD$ is measured by $\frac{1}{2}\,\overset{\frown}{AD}$. (Case I.)

and $\angle BCD$ is measured by $\frac{1}{2}\,\overset{\frown}{BD}$. (Case I.)

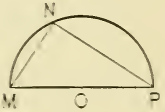
$\therefore \angle ACB$ is measured by $\frac{1}{2}\,\overset{\frown}{AB}$. (Ax. 3.)



**233.** COR. 1. *Angles inscribed in the same segment are equal.*

($\angle A = \angle B = \angle C = \angle D$).

**234.** COR. 2. *An angle inscribed in a semicircle is a right angle* ($\angle N = 90°$).



NOTE. The statement "$\angle AOB$ is measured by $\overset{\frown}{AB}$" is a brief form for writing an equation, viz. "the numerical measure of $\angle AOB$ = the numerical measure of $\overset{\frown}{AB}$." Hence, this and similar statements may be treated like equations, *e.g.* the axioms 1, 2, and 8 may be applied to them.

**235.** COR. 3. *The opposite angles of an inscribed quadrilateral are supplementary.*

---

Ex. **1.** If in the diagram for Case I, $\angle C = 30°$, how many degrees are in $\overset{\frown}{CB}$?

Ex. **2.** If in the same diagram $\overset{\frown}{BC} = 3\,\overset{\frown}{AB}$, find $\angle C$.

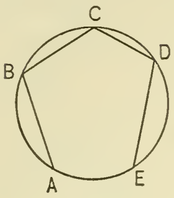
Ex. **3.** If in the diagram for Case II $\overset{\frown}{AC}$ is $\frac{1}{3}$, and $\overset{\frown}{BC}$ is $\frac{1}{4}$ of the circle, find $\angle ACB$, $\angle ACD$.

Ex. **4.** If in the diagram for Case III, $A$ is the mid-point of $\overset{\frown}{CD}$, and $B$ is the mid-point of $\overset{\frown}{AD}$, how many degrees are there in $\angle ACB$?

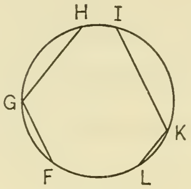
Ex. **5.** If a quadrilateral $ABCD$ be inscribed in a circle, and the two diagonals be drawn, find all the angles in the figure, if $\overset{\frown}{AB} = 80°$, $\overset{\frown}{BC} = 110°$, and $\overset{\frown}{CD} = 90°$.

Ex. **6**. In a diagram of the preceding exercise, find four pairs of equal angles.

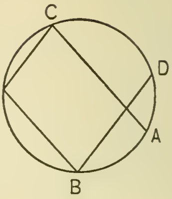
Ex. **7**. Find $\angle B + \angle D$, if $\widehat{AE} = 40°$.



Ex. 7



Ex. 8



Ex. 9

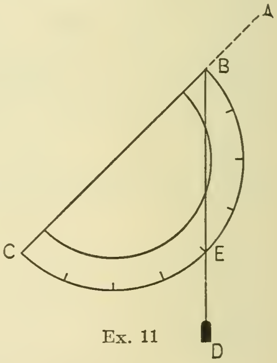
Ex. **8**. How many degrees are in $\angle G + \angle K$, if $\widehat{HI} = 20°$, and $\widehat{FL} = 30°$?

Ex. **9**. How many degrees are in $\angle B + \angle C$, if $\widehat{DA} = 50°$?

Ex. **10**. If through one of the points of intersection of two equal circles a line be drawn to meet the circles, the extremities of that line are equidistant from the other point of intersection.

Ex. **11**. If a semicircle is placed so that the prolongation of its diameter $CB$ passes through a point $A$, and a plumb line $BD$ intersects the semicircle at $E$, the angle of elevation of $A$ is measured by $\frac{1}{2}\widehat{BE}$.

[For the definitions of angle of elevation and additional practical applications, see problems 34–36, p. 287.]



Ex. 11

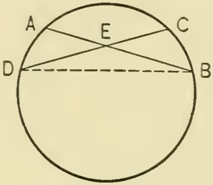
Ex. **12**. A circle is divided into four parts in the continued ratio of 2 : 4 : 4 : 5. The points of division are joined in succession. Find the number of degrees in the inscribed angles thus formed.

Ex. **13**. Lines are drawn on the face of a clock from 1 to 5 to 7 to 11 to 1. Find the number of degrees in each inscribed angle.

Ex. **14**. An equilateral hexagon is inscribed in a circle. How many degrees are there in each angle of the hexagon?

### PROPOSITION XV. THEOREM

**236.** *An angle formed by two chords intersecting within the circle is measured by one half the sum of its intercepted arc and the arc of its vertical angle.*



**Given** two chords $AB$ and $CD$ intersecting in $E$ in $\odot O$.

**To prove** $\angle AED$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{CB})$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $DB$. | **1.** Why? |
| **2.** $\angle AED = \angle D + \angle B$. | **2.** Ext. $\angle$ of $\triangle = \cdots$. |
| **3.** $\angle D$ is meas. by $\frac{1}{2}\overset{\frown}{CB}$.<br>$\angle B$ is meas. by $\frac{1}{2}\overset{\frown}{AD}$. | **3.** Inscr. $\measuredangle$ $\cdots$. |
| **4.** $\angle D + \angle B$ is meas. by $\frac{1}{2}(\overset{\frown}{CB} + \overset{\frown}{AD})$. | **4.** Why? |
| **5.** $\therefore$ $\angle AED$ is meas. by $\frac{1}{2}(\overset{\frown}{CB} + \overset{\frown}{AD})$. | **5.** Why? |

---

Ex. **1.** If in diagram for Prop. XV, $\overset{\frown}{AD} = 60^\circ$, $\overset{\frown}{BA} = 140^\circ$, and $\overset{\frown}{CB} = 20^\circ$, find $\angle AEC$ and $\angle DAE$.

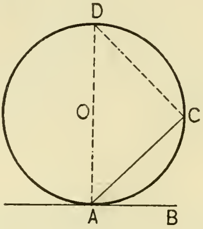
Ex. **2.** If two perpendicular chords intersect within the circle, the sum of a pair of opposite intercepted arcs is equal to a semicircle.

Ex. **3.** If in diagram for Prop. XV, $\overset{\frown}{AC}$ and $\overset{\frown}{DB}$ add to 210°, find $\angle CEB$.

Ex. **4.** In same diagram (Prop. XV) $\angle B = 40^\circ$, and $\overset{\frown}{BC} = 100^\circ$. Find $\angle AED$.

## PROPOSITION XVI. THEOREM

**237.** *An angle formed by a tangent to a circle and a chord drawn from the point of contact is measured by half the intercepted arc.*



**Given** $AB$ a tangent, and $AC$ a chord in $\odot O$.

**To prove** $\angle CAB$ is measured by $\frac{1}{2}\widehat{AC}$.

### Proof

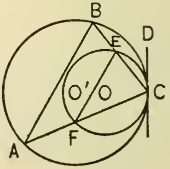
| STATEMENTS | REASONS |
|---|---|
| **1.** Draw the diameter $AD$ and chord $DC$. | **1.** Why? |
| **2.** $\angle C$ is a rt. $\angle$. | **2.** (234). |
| **3.** $\angle DAB$ is a rt. $\angle$. | **3.** (204). |
| **4.** $\angle D$ is compl. to $\angle DAC$. | **4.** (112). |
| **5.** $\angle BAC$ is compl. to $\angle DAC$. | **5.** Why? |
| **6.** $\angle BAC = \angle D$. | **6.** Why? |
| **7.** $\angle D$ is meas. by $\frac{1}{2}\widehat{AC}$. | **7.** Why? |
| **8.** $\therefore\ \angle BAC$ is meas. by $\frac{1}{2}\widehat{AC}$. | **8.** Equal ∡ have same measure. |

Ex. **1.** If, in diagram for Prop. XVI, $\widehat{AC} = 2\widehat{CD}$, find $\angle CAB$.

Ex. **2.** A chord is parallel to the tangent drawn through the mid-point of the subtended arc.

Ex. **3.** Circles $O$ and $O'$ are tangent to $DC$ at $C$. Prove that

$$\angle DCB = \angle A = \angle EFC.$$

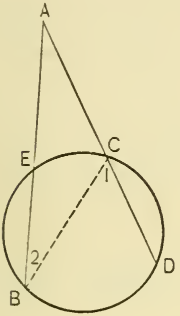


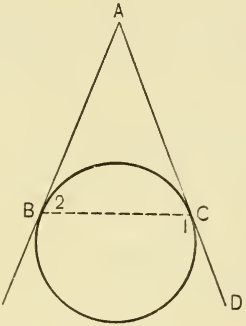
Ex. 3

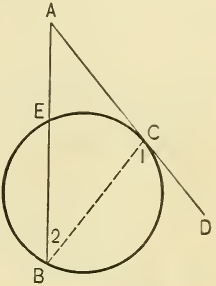
Ex. **4.** Two tangents to a circle form an angle of 40°. How many degrees are there in the minor arc between the tangents?

## PROPOSITION XVII. THEOREM

**238.** *An angle formed by two secants, or two tangents, or a tangent and a secant, intersecting without a circle, is measured by half the difference between the intercepted arcs.*







CASE I. **Given** $AB$ and $AD$ secants drawn from an external point $A$ to $\odot O$, cutting circle at $E$ and $C$.

**To prove** $\angle A$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{EC})$.

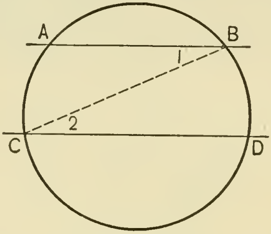
### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $BC$. | **1.** Why? |
| **2.** $\angle A + \angle 2 = \angle 1$. | **2.** Why? |
| **3.** $\angle A = \angle 1 - \angle 2$. | **3.** Why? |
| **4.** $\angle 1$ is meas. by $\frac{1}{2}\ \overset{\frown}{BD}$. | **4.** Ins. $\angle$ is meas. by···. |
| **5.** $\angle 2$ is meas. by $\frac{1}{2}\ \overset{\frown}{EC}$. | **5.** Same as 4. |
| **6.** $\angle 1 - \angle 2$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{EC})$. | **6.** Diff. of 2 $\measuredangle$ is meas. by diff. of their measures. |
| **7.** $\therefore\ \angle A$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{EC})$. | **7.** Sub. |

NOTE. Let the student prove Cases II and III.

## PROPOSITION XVIII. THEOREM

**239.** *Parallel secants intercept equal arcs on a circle.*



**Given** $AB$ and $CD$ two parallel secants to $\odot O$.

**To prove** $\overset{\frown}{AC} = \overset{\frown}{BD}$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $CB$. | **1.** Why? |
| **2.** $\angle 1$ is meas. by $\frac{1}{2}\ \overset{\frown}{AC}$.<br>$\angle 2$ is meas. by $\frac{1}{2}\ \overset{\frown}{BD}$. | **2.** Ins. $\angle$ is meas. by $\frac{1}{2}$ its arc. |
| **3.** $AB \parallel CD$. | **3.** Hyp. |
| **4.** $\angle 1 = \angle 2$. | **4.** Alt. int. $\angle$s of $\parallel$s are $=$. |
| **5.** $\frac{1}{2}\ \overset{\frown}{AC} = \frac{1}{2}\ \overset{\frown}{BD}$. | **5.** Equal $\angle$s have $=$ measures. |
| **6.** $\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$. | **6.** Why? |

**240.** COR. 1. *If a secant is parallel to a tangent, they intercept equal arcs on the circle.*

**241.** COR. 2. *Two parallel tangents intercept equal arcs on the circle.*

---

Ex. **1.** If the angle formed by two tangents is 60°, how many degrees are in each of the intercepted arcs?

Ex. **2.** If an angle formed by a secant and a tangent is 20° and the greater of the intercepted arcs is 90°, how many degrees are in the other intercepted arc?

Ex. **3**. If in the diagram for Prop. XVII, Case I. $\widehat{BD} = 100°$ and $\angle A = 20°$, find $\widehat{EC}$.

Ex. **4**. If in the same diagram $\widehat{EC} = 60°$ and $\widehat{EB} = \widehat{BD} = \widehat{CD}$, find $\angle A$.

Ex. **5**. In the diagram for Prop. XVIII, find $\angle ABC$ if $\widehat{AB} = 80°$ and $\widehat{CD} = 120°$.

Ex. **6**. The line bisecting an angle formed by a tangent and a chord drawn to the point of contact, bisects the intercepted arc.

Ex. **7**. *An inscribed trapezoid is isosceles.*

Ex. **8**. In diagram for Prop. XVIII, if $\widehat{AB} = 80°$ and $\widehat{CD} = 200°$, find $\angle B$.

Ex. **9**. An exterior angle of an inscribed quadrilateral is equal to its opposite interior angle.

Ex. **10**. Find the sum of three alternate angles of an inscribed hexagon.

Ex. **11**. If in the greater of two concentric circles, chords be drawn touching the smaller circle, the chords are equal.

Ex. **12**. If two equal chords intersect, the lines joining their ends form an isosceles trapezoid.

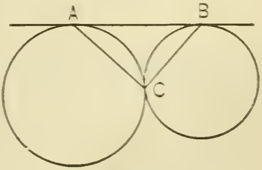
Ex. **13**. If the bisector of an inscribed angle be produced until it meets the circle, and through this point of intersection a chord be drawn parallel to one side of the angle, it is equal to the other side.

Ex. **14**. A circle constructed on an arm of an isosceles triangle as diameter bisects the base.

Ex. **15**. If two circles intersect in $A$ and $B$ and $AC$ and $AD$ are diameters of the two circles, then a line joining $C$ and $D$ passes through $B$.

Ex. **16**. If two circles are tangent to each other and through the point of contact two lines are drawn, each terminating in the two circles, the chords joining the ends of these lines are parallel.

Ex. **17**. If two circles are tangent at $C$ and a common exterior tangent touches the circles in $A$ and $B$, the angle $ACB$ is a right angle.



Ex. 17

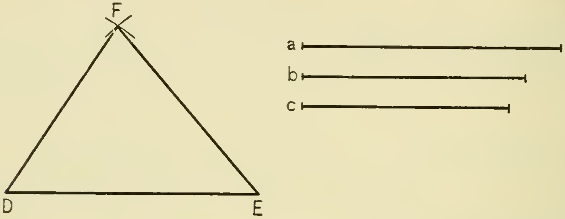
* Ex. **18**. If from the extremities of a diameter perpendiculars be drawn upon any chord (produced, if necessary), the feet of the perpendiculars are equidistant from the center.

## CONSTRUCTIONS

**242.** Note. In the following examples, we shall denote the given parts of a triangle always in the same manner; the sides by $a$, $b$, $c$, the opposite angles by $A$, $B$, and $C$, the altitudes by $h_a$, $h_b$, and $h_c$, the medians by $m_a$, $m_b$, and $m_c$, and the angle-bisectors by $t_a$, $t_b$, and $t_c$.

### Proposition XIX. Problem

**243.** *To construct a triangle, having given the three sides.* (*s. s. s.*)



**Given** $a$, $b$, and $c$ the three sides of a △.

**Required** to construct the triangle.

**Construction.** Draw $DE = a$.

From $E$ as a center, with a radius equal to $b$, draw an arc.

From $D$ as a center, with a radius equal to $c$, draw an arc.

The arcs intersect at $F$. Draw $FE$ and $FD$.

△$DEF$ is the required triangle.

Discussion. The construction is impossible if one side is greater than, or equal to, the sum of the other two.

Ex. **1.** Construct an equilateral triangle, having one side given.

Ex. **2.** *Construct an equilateral triangle whose perimeter equals a given line.*

Ex. **3.** Construct a quadrilateral, having given the four sides and one diagonal.

### PROPOSITION XX. PROBLEM

**244.** *To construct a triangle, having given two angles and the included side.* (*a. s. a.*)

The solution is left to the student.

---

Ex. **1.** *Upon a given arm, to construct an isosceles right triangle.*

Ex. **2.** Construct an isosceles triangle, having given the base and a base angle.

**245.** REMARK. **The possibility of a solution of a problem depends often upon the proper choice of the part which is drawn first.**

### PROPOSITION XXI. PROBLEM

**246.** *To construct a triangle, having given two sides and the included angle.* (*s. a. s.*)

The solution is left to the student.

---

Ex. **1.** Construct an isosceles triangle, having given an arm and the vertex angle.

Ex. **2.** *Construct a right triangle, having given the two arms.*

Ex. **3.** To construct a quadrilateral ($ABCD$), having given three sides ($AB$, $BC$, and $CD$), the angle formed by the last of these sides and the unknown side ($\angle D$), and the angle formed by the same side and a diagonal ($\angle DCA$).

Ex. **4.** To construct a pentagon ($ABCDE$), having given the five sides ($AB$, $BC$, $CD$, $DE$, and $EA$) and two angles including one of these sides ($\angle A$ and $\angle B$).

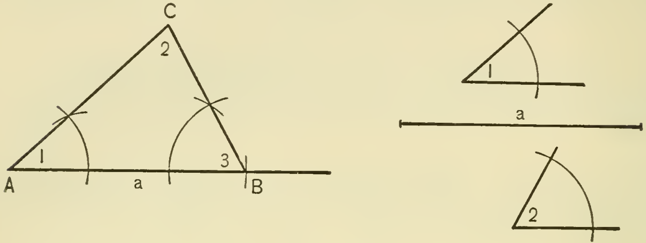
Ex. **5.** Construct a quadrilateral, having given the four sides and one angle.

Is it possible to solve exercise five by constructing first a side not adjacent to the given angle?

Ex. **6.** *To construct a rhombus having given the perimeter and one angle.*

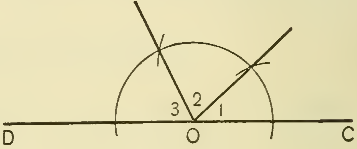
### Proposition XXII. Problem

**247.** *To construct a triangle, having given one side, one adjacent and one opposite angle.* (*s. a. a.*)



**Given** ∠1 and ∠2 and side *a* opposite ∠2.

**Required** to construct the △.



**Construction.** In straight ∠*DOC* at *O* lay off ∡ 1, 2, as above. The remaining ∠3 is equal to the third ∠ of the triangle.

The required △ is *ABC* and is constructed by the method of Prop. XX. (a. s. a.)

Discussion. The construction is impossible if the sum of the given angles is greater than or equal to a straight angle.

---

Ex. **1.** Construct by means of Prop. XXII a right triangle, having given the hypotenuse and an acute angle.

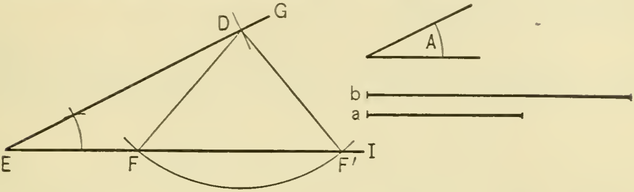
Ex. **2.** Find a construction of the same problem which does not depend upon Prop. XXII.

Ex. **3.** *Construct a right triangle, having given an arm and the opposite angle.*

Ex. **4.** Construct an equilateral triangle, having given
- (*a*) *the altitude.*
- (*b*) the radius of the inscribed circle.
- (*c*) the radius of the circumscribed circle.

### PROPOSITION XXIII. PROBLEM

**248.** *To construct a triangle, having given two sides and an angle opposite one of them.*



**Given** $a$ and $b$, two sides of a △, and ∠$A$ opposite side $a$.

**Required** to construct the △.

**Construction.** Draw ∠$GEI$ = ∠$A$.

On $EG$, lay off $ED = b$.

From $D$ as a center with a radius equal to $a$, draw an arc intersecting $EI$ in $F$ and $F'$.

Both ⚠$EDF$ and $EDF'$ fulfill the required conditions.

DISCUSSION. If the arc intersects the base line twice on the same side of point $E$, there are two solutions, and if it touches the line or intersects once, there is one solution. If it does not touch the line, a solution is impossible.

---

Ex. **1.** In Prop. XXIII, how many solutions are possible, when angle $A$ is obtuse? Right? Acute?

Ex. **2.** *Construct a right triangle, having given the hypotenuse and one arm.*

Ex. **3.** Solve the preceding problem by drawing the hypotenuse first.

**249.** A triangle may be constructed if the following parts are given:

(1) Three sides.

(2) Two sides and the included angle.

(3) Two angles and the included side.

(4) Two angles and a non-included side.

(5) Two sides and the angle opposite one of them.

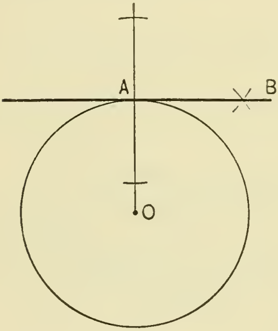
To construct a triangle, three parts must be given, one part always being a side.

---

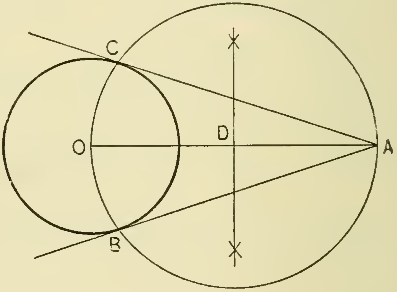
Ex. Are the three angles of a triangle three independent parts, and can a triangle be constructed when the three angles are given?

### PROPOSITION XXIV. PROBLEM

**250.** *From a given point, to draw a tangent to a given circle.*



CASE I



CASE II

I. When the given point, *A*, is in the circle.

HINT. What is the angle formed by a radius and a tangent at its extremity?

II. When the given point, *A*, is without the circle.

**Construction.** Join *A* to *O*, the center of the given circle. From *D*, the mid-pt. of *OA*, as center and *DA* as a radius construct a circle intersecting the given circle in *B* and *C*.

Then *AC* and *AB* are the required tangents.

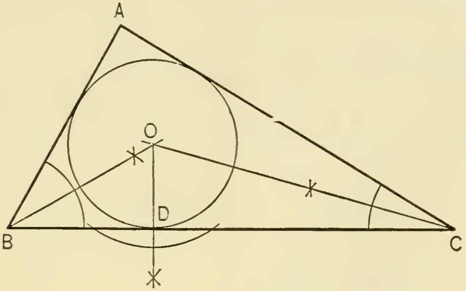
HINT. Show that ∡ *ACO* and *OBA* are right angles.

Ex. **1.** Construct a line tangent to a given circle and parallel to a given line.

Ex. **2.** Construct a line tangent to a given circle and perpendicular to a given line.

### Proposition XXV. Problem

**251.** *To inscribe a circle in a given triangle.*



**Given** $\triangle ABC$.

**Required** to inscribe a circle in $\triangle ABC$.

**Construction.** Construct the bisectors of $\angle B$ and $\angle C$.

These bisectors meet at some point $O$. (160)

From $O$ draw $OD \perp BC$.

From $O$ as a center, with a radius equal to $OD$, draw a circle, which is the required one.

[The proof is left to the student. See (160) and (203).]

**252.** Def. A circle touching one side of a triangle and the prolongations of the other two sides is an **escribed circle.**

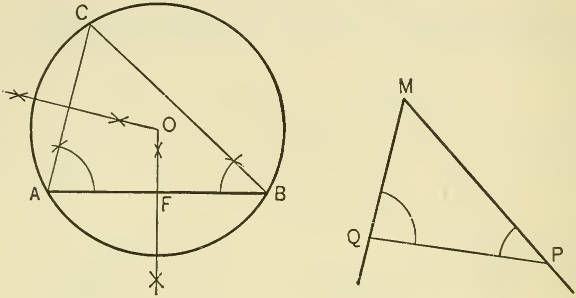
Ex. **1.** Construct the three escribed circles of a triangle.

Ex. **2.** The bisector of an angle of a triangle meets the circumscribed circle in a point which is equidistant from the other two vertices of the triangle and the center of the inscribed circle.

Ex. **3.** A circle is inscribed in a triangle whose sides are 9, 8, and 9. Find the distances of the vertices from the points of contact (209).

### PROPOSITION XXVI. PROBLEM

**253.** *Upon a given straight line as chord, to construct a segment of a circle which shall contain an angle equal to a given angle.*



**Given** line $AB$ and $\angle M$.

**Required** to construct a segment of a circle on $AB$ as chord which shall contain an angle equal to $\angle M$.

**Construction.** In the sides of $\angle M$ take any two points $P$ and $Q$ respectively.

Draw $PQ$.

On $AB$ as a base construct $\triangle ABC$ so that $\angle A = \angle Q$, and $\angle B = \angle P$.

Circumscribe a circle about $\triangle ABC$.

Segment $ACB$ is the required segment.

**Proof.** $\angle C = \angle M$. (113)

$\therefore$ Any angle inscribed in the segment $= \angle M$. (233)

---

Ex. On a line 2″ long construct a segment of a circle which shall contain an angle of 30°; an angle of 45°; an angle of 60°; an angle of 135°.

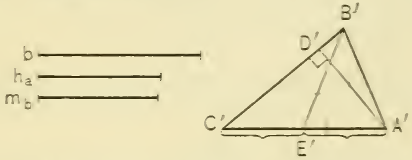
## ANALYSIS OF PROBLEMS

**254.** An analysis of a problem is a course of reasoning by which its construction is discovered. Although no rules can be given which apply to all constructions, the method explained in the following exercises may be used in many problems.

Ex. A. *To construct a triangle having given one side, the corresponding median, and the altitude to another side.*

**Given** $b$, one side of triangle, $m_b$ the corresponding median, and $h_a$ the altitude upon another side.

**Required** to construct the △.



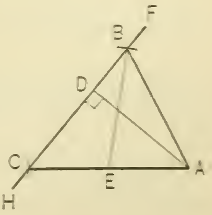
**Analysis.** (1) Suppose $A'B'C'$ were the required triangle.

(2) Then we should know $C'A'$ ($= b$), $C'E'$ and $E'A' = \frac{b}{2}$, $A'D'$ ($= h_a$), $B'E'$ ($= m_b$), and ∠ $A'D'C'$ and $A'D'B'$ ($=$ rt. ∠).

(The student is advised to mark the known parts or to draw them in a different color from the other lines.)

(3) Examine all triangles in the figure, and see if one can be constructed. The rt. △$A'D'C'$ can be constructed, having given two sides.

(4) Make this triangle the basis of the construction. Hence,



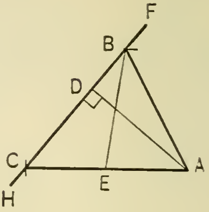
**Construction.** Draw $DA = h_a$.

At $D$, draw $FH \perp AD$.

From $A$ as a center, with a radius equal to $b$, draw an arc intersecting $DH$ in $C$. Draw $AC$.

Bisect $AC$ in $E$, and from $E$ as a center, with a radius equal to $m_b$, describe an arc, meeting $FH$ in $B$.

$ABC$ is the required triangle.



**Proof.** $AD = h_a$.

$BE = m_b$.

$AC = b$.

$AD$ is an altitude, as

$\angle ADC =$ rt. $\angle$.

$CE = EA$

$\therefore EB$ is a median.

NOTE. The student should first carefully note whether the parts given are such as to make the construction possible. In Ex. A, the construction is impossible if $AD > AC$.

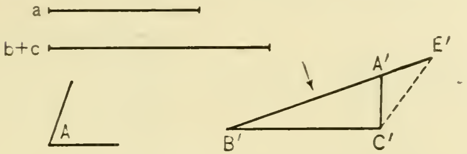
**255.** The following rules express the procedure in a general form:

1. Make a diagram resembling the one required, but not necessarily having the same dimensions.
2. Determine (*a*) all lines, (*b*) all angles, that are directly given, or that can be easily found from the given parts, and mark these parts.
3. Examine all triangles of the diagram until you discover a triangle that can be constructed.
4. Make this triangle the basis of the construction, and try to determine successively all other parts of the figure.
5. In case no triangle can be found that can be constructed directly, draw additional lines which will enable you to obtain such a triangle.

**256.** Some problems require the drawing of additional lines (as stated above under 5 in section 255). In particular:

**When a sum or a difference is given, construct such sum or difference in the analysis.**

Ex. B. *To construct a triangle, having given the base, the sum of the other two sides, and the angle included by the two.*



**Analysis.** 1. Suppose $A'B'C'$ were the required triangle.

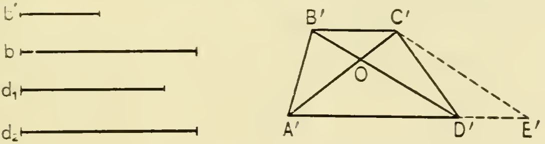
2. Then we should know two parts only.

$\therefore$ produce $BA'$ to $E'$ so that $A'E' = A'C'$.

Draw $E'C'$. We now know $C'B'$ $(=a)$, $B'E'(=b+c)$, $\angle B'A'C'(=\angle A)$, $\angle E'\left(=\frac{\angle A}{2}\right)$, and $\angle A'C'E'\left(=\frac{\angle A}{2}\right)$.

3. Examine all triangles. As $\triangle B'C'E'$ can be constructed, make this the basis of the construction.

Ex. C. *To construct a trapezoid, having given each base* ($b$ and $b'$) *and each diagonal* ($d_1$ and $d_2$).



**Analysis.** 1. Suppose $A'B'C'D'$ were the required trapezoid. Produce $A'D'$ to $E'$ so that $D'E' = B'C'$.

Draw $C'E'$.

2. We know $A'E' = b + b'$, $A'C' = d_1$, $B'D' = C'E' = d_2$.

3. $\triangle A'C'E'$ can be constructed (s. s. s.).

4. $\square B'C'E'D'$ can now be constructed since we know two sides and the included angle.

5. $A'$, $B'$, $C'$, $D'$, the four vertices, are found so the trapezoid can be completed.

Construct a triangle, having given:

**Ex. 1.** $a, b, m_b$.

**Ex. 2.** $a, b, h_b$.

**Ex. 3.** $b, h_a, \angle A$.

**Ex. 4.** $b, t_c, \angle A$.

**Ex. 5.** $a, h_b, \angle B$.

**Ex. 6.** $a, b, h_c$ (2 $\triangle$).

**Ex. 7.** $\angle C, t_c, b$.

**Ex. 8.** $\angle A, \angle B, h_a$.

**Ex. 9.** $a, h_a, m_a$.

**Ex. 10.** $b, h_a, m_a$ (2 $\triangle$).

**Ex. 11.** $a, m_a, m_b$.

**Ex. 12.** $a, h_b, h_c$.

**Ex. 13.** $a, b, \angle A + \angle B$.

**Ex. 14.** $a, \angle B, b + c$.

Ex. **15.** *Construct an isosceles triangle, having given the base and the vertex angle.*

Ex. **16.** Construct an isosceles triangle, having given the sum of base and an arm, and a base angle.

Ex. **17.** Construct a triangle, having given an angle, an adjacent side, and the difference of the other two sides.

Ex. **18.** Construct a triangle, having given the base, the difference of the other two sides, and the angle included by the two sides.

Construct a right triangle, having given:

Ex. **19.** *One arm and the altitude upon the hypotenuse.*

Ex. **20.** The hypotenuse and the difference between the arms.

Ex. **21.** The hypotenuse and the sum of the arms.

Ex. **22.** *Construct a parallelogram, having given the sides and a diagonal.*

## LOCI

**257. A locus of a point** in a plane is a line or a group of lines, such that ($a$) all points in the line (or lines) satisfy a certain condition, and ($b$) all points satisfying the condition are in the line (or lines).

Thus ($a$) Every point in the perpendicular-bisector of a line is equidistant from the ends of a line. (125)

($b$) Every point equidistant from the ends of a line lies in the perpendicular-bisector of the line. (80).

$\therefore$ The perpendicular-bisector is the *locus* of the point that is equidistant from the ends of the line.

**258.** The word *locus* (plural *loci*) means "place," and it signifies in geometry the place of a point that moves according to certain conditions.

Thus, if one end of a stretched cord is fixed upon a drawing board, the other end, when moved about, will describe a circle. The circle is the locus of the other end.

Every point of an elevator car moving up or down moves in a straight line, which is perpendicular to the ground. Such a straight line is the locus of the moving point.

Similarly the locus of a point of the minute hand of a watch is a circle, etc.

**259.** To prove that a certain line is a locus, we must establish that:

(1) **Every point in the line or group of lines satisfies the given condition; and,**

(2) **Every point that satisfies the condition lies in the line or group of lines.** This is done by proving the theorem corresponding to (1) and by proving its converse corresponding to (2).

Note. We may establish (1) and (2′).

(2′) Every point outside the line or group of lines does *not* satisfy the given condition.

Thus, to find the locus of a point at a distance of $\frac{3}{5}$ inch from $AB$, it is not sufficient to prove that every point in $CD$ has the proper distance from $AB$. It is also necessary to show that no point without satisfies the condition, which leads here to the discovery of another part of the locus, $EF$, located on the other side of $AB$.

Ex. Draw the following loci without giving proofs:

(*a*) The locus of a point $\frac{3}{4}$ in. from a given point.

(*b*) The locus of a point $\frac{1}{4}$ in. from a given line.

(*c*) The locus of a point equidistant from a pair of parallel lines.

(*d*) The locus of the mid-point of all chords drawn in a given circle and parallel to a given line.

(*e*) The locus of the mid-points of all lines that are parallel to the base of a triangle and terminated by the other sides.

(*f*) The locus of a point equidistant from two given lines.

(*g*) The locus of the end of a line tangent to a given circle and $\frac{1}{4}$ in. long.

(*h*) The locus of the center of a circle whose radius equals $\frac{1}{4}$ in., and that touches a given circle (radius 1 in.) externally.

## LOCUS THEOREMS

**260.** The locus of a point at a given distance from a given point is the circle described from the point as center with the given distance as radius.

For (*a*) every point in the circle has the required distance, while (*b*) every point at the required distance is in the circle.

**261.** The locus of a point equidistant from the ends of a given line is the perpendicular-bisector of that line.

For (*a*) every point in the perpendicular-bisector is equidistant. (*b*) Every point equidistant from the ends of the line lies in the perpendicular-bisector. (80)

**262.** The locus of a point that is at a given distance from a given straight line consists of two lines parallel to the given line at the given distance.

**263.** The locus of a point equidistant from two given parallel lines is a third parallel lying midway between the two given parallel lines.

**264.** The locus of a point equidistant from two intersecting straight lines consists of the bisectors of the included angles.

---

Ex. **1.** Find the locus of the mid-points of the radii of a given circle.

Ex. **2.** *Find the locus of the vertex of all right angles whose sides pass through two given points.*

Ex. **3.** *Find the locus of the center of a circle which has a given radius and touches a given line.*

Ex. **4**. Find the locus of the center of a circle which has a given radius and passes through a given point.

Ex. **5**. *Find the locus of the center of a circle that passes through two given points.*

Ex. **6**. *Find the locus of the center of a circle that touches two given lines.*

Ex. **7**. Find the locus of the center of a circle which has a given radius and touches a given circle.

Ex. **8**. *Find the locus of the center of a circle touching a given line at a given point.*

Ex. **9**. *Find the locus of the center of a circle that touches a given circle in a given point.*

Ex. **10**. Two vertices, $B$ and $C$, of a triangle have a fixed position. Find the locus of the third vertex ($A$) if $h_a$ equals a given line.

Ex. **11**. Two vertices, $B$ and $C$, of a triangle have a fixed position. Find the locus of the third vertex ($A$) if $m_a$ equals a given line.

Ex. **12**. The base of a parallelogram has a fixed position and length, and the adjacent side has a given length. Find the locus of the intersection of the diagonals.

Ex. **13**. The base of a rectangle has a fixed position. Find the locus of the intersection of the diagonals.

Ex. **14**. *Find the locus of the mid-points of all chords that have a given length and are drawn in a given circle.*

Ex. **15**. Find the locus of the mid-points of all chords that can be drawn from a given point in the circle.

Ex. **16**. **The locus of the vertex of a triangle which has a given (fixed) base and a given vertex angle is an arc** (constructed according to 253, p. 140).

**265**. Loci are used to determine the position of a point (or points) that satisfies two conditions. Each condition determines a locus, and the point (or points) of intersection of the two loci is the required point.

In the following exercises, state under what conditions no point, one point, or several points may be found.

Ex. **1**. In a given line, $AB$, find a point at a given distance, $d$, from a given point $C$.

Ex. **2**. In a given line, $AB$, find a point at a given distance, $d$, from a given line $CD$.

Ex. **3**. In a given line, $AB$, find a point equidistant from two given points, $P$ and $Q$.

Ex. **4**. In a given circle, find a point at a given distance, $d$, from a given point $C$.

Ex. **5**. In a given circle, find a point equidistant from two given parallel lines, $CD$ and $EF$.

Ex. **6**. In a given circle, find a point equidistant from two given intersecting lines, $CD$ and $EF$.

Ex. **7**. Find a point equidistant from two given intersecting lines, $AB$ and $CD$, and at a given distance from a given point, $E$.

Ex. **8**. Find a point equidistant from two given intersecting lines, $AB$ and $CD$, and at a given distance from a given line, $EF$.

Ex. **9**. Find a point equidistant from two given intersecting lines, $AB$ and $CD$, and equidistant from two given points, $E$ and $F$.

Ex. **10**. Find a point equidistant from two given points, and having a given distance from a given point, $E$.

Ex. **11**. Find a point equidistant from two given points and equidistant from two given parallel lines, $EF$ and $GH$.

Ex. **12**. Find a point equidistant from two given parallel lines and equidistant from two given intersecting lines, $EF$ and $GH$.

Ex. **13**. Find a point at a given distance $d$, from a given line, $AB$, and equidistant from two given points, $E$ and $F$.

Ex. **14**. Find a point having a given distance, $d$, from a given line, $AB$, and equidistant from two given parallel lines, $EF$ and $GH$.

To construct a circle having a given radius:

Ex. **15**. Touching a given line and passing through a given point.

Ex. **16**. Touching two given lines.

Ex. **17**. Passing through a point and touching a given circle.

Ex. **18**. Touching two given circles.

Ex. **19**. Touching a given circle and a given line.

To construct a circle:

Ex. **20**. Touching a given line in a given point and passing through another point without.

Ex. **21**. Touching a given circle at a given point and passing through another given point without.

Ex. **22**. Touching a given line and a given circle at a given point.

Hint. Draw a tangent to the circle at the given point.

* Ex. **23**. Construct a triangle having given $a$, $h_a$, $\angle A$.

Ex. **24**. Construct a triangle having given $a$, $m_a$, $\angle A$.

**266**. No general method can be given for the solution of exercises; a great many, however, can be solved:

(1) **By a gradual putting together of the given parts.** (245)

(2) **By means of an analysis.**

(3) **By means of loci.**

## MISCELLANEOUS EXERCISES

Construct an isosceles triangle, having given:

Ex. **1**. The base and the altitude upon an arm.

Ex. **2**. The altitude upon the base and the vertex angle.

Ex. **3**. The vertex angle and the sum of one arm and the base.

Ex. **4**. The perimeter and the base angles.

Construct a right triangle, having given:

Ex. **5**. One acute angle and the altitude upon the hypotenuse.

Ex. **6**. The altitude upon the hypotenuse and one of the segments of the hypotenuse.

Ex. **7**. The sum of the arms and one acute angle.

Ex. **8**. To find a point in one side of a triangle which is equidistant from the other two sides.

Ex. **9**. *Find the locus of the vertex of a right triangle, having a given hypotenuse.*

Ex. **10**. In one side of a quadrilateral to find a point equidistant from the ends of the opposite side.

Ex. **11**. From a point $P$, in a circle, to draw a chord, having a given distance from the center.

Ex. **12**. In a given circle, to draw a diameter having a given distance from a given point.

Ex. **13**. Through a given point to draw a line, having a given distance from another point.

Ex. **14**. Through two given points in a circle, to draw two equal parallel chords.

Ex. **15**. *Trisect a given straight angle.*

Ex. **16**. *Trisect a given right angle.*

Ex. **17**. *Trisect a semicircle.*

Ex. **18**. Through a given point outside a given line, to draw a line making a given angle with a given line.

Ex. **19**. Through a given point, to draw a line of given length terminating in two given parallel lines.

Ex. **20**. Through a given point, to draw a line making equal angles with two given lines.

* Ex. **21**. To bisect an angle formed by two lines, without producing them to their intersection.

To construct a triangle, having given: (Note 242)

Ex. **22**. $a$, $\angle B$, $h_a$.

Ex. **23**. $a$, $\angle B$, $m_c$.

Ex. **24**. $a$, $\angle B$, $t_c$.

Ex. **25**. $\angle A$, $h_a$, $t_a$.

Ex. **26**. $a + b + c$, $\angle B$, $\angle C$.

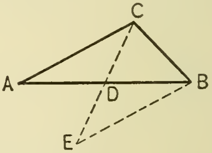
Ex. **27**. $a$, $b$, $R$.[1]

Ex. **28**. $a$, $h_a$, $R$.[1]

Ex. **29**. $h_a$, $h_b$, $\angle B$.

Ex. **30**. $a$, $b$, $m_c$.

HINT. Produce $m_c$ its own length.



Ex. 30

* Ex. **31**. $a$, $m_c$, $\angle C$.

* Ex. **32**. $m_a$, $m_b$, $h_c$.

* Ex. **33**. $m_a$, $h_b$, $h_c$.

* Ex. **34**. $m_a$. $m_b$, $m_c$.

To construct a square, having given:

Ex. **35**. *The diagonal.*

Ex. **36**. The difference between the diagonal and the side.

Ex. **37**. The sum of the diagonal and the side.

To construct a rectangle, having given:

Ex. **38**. *One side and a diagonal.*

Ex. **39**. *One side and the angle formed by the diagonals.*

Ex. **40**. The perimeter and the diagonal.

---

[1] $R$ = radius of circumscribed circle.

To construct a rhombus, having given:

Ex. **41**. *The two diagonals.*

Ex. **42**. The perimeter and one diagonal.

Ex. **43**. *One angle and a diagonal.*

Ex. **44**. *The altitude and the base.*

Ex. **45**. The altitude and one angle.

To construct a parallelogram, having given:

Ex. **46**. *Two adjacent sides and one altitude.*

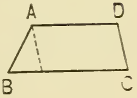
Ex. **47**. *Two adjacent sides and an angle.*

Ex. **48**. *One side and two diagonals.*

Ex. **49**. One side, one angle, and one diagonal.

Ex. **50**. *The diagonals and the angle formed by the diagonals.*

**267**. In the analysis of a problem relating to a trapezoid, draw a line through one vertex, *A*, either parallel to the opposite arm, *DC*, or parallel to a diagonal, *DB*.



To construct a trapezoid, having given:

Ex. **1**. *The four sides.*

Ex. **2**. *The bases and the lower base angles.*

Ex. **3**. The bases, another side, and one base angle.

Ex. **4**. The bases and the diagonals.

Ex. **5**. One base, the diagonals, and the angle formed by the diagonals.

Ex. **6**. *To draw a common external tangent to two given circles.*

Ex. **7**. *To draw a common internal tangent to two given circles.*

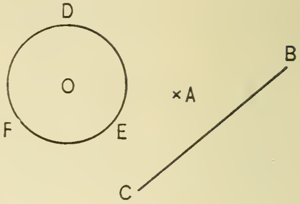
Ex. **8**. About a given circle, to circumscribe a triangle, having given the angles.

Ex. **9**. Find the locus of the mid-points of the secants that pass through a given point without a circle.

Ex. **10**. In a given circle, to inscribe a triangle, having given the angles.

* Ex. **11**. From a given point in a circle, to draw a chord that is bisected by a given chord.

Ex. **12**. Given a point, $A$, between a circle and a straight line. Through $A$, to draw a line terminated by the circle and the given line, and bisected in $A$.



Ex. 12

Ex. **13**. Given two points, $A$ and $B$, on the same side of a line, $CD$. To find a point, $X$, in $CD$, such that $\angle AXC = \angle BXD$.

Ex. **14**. If two circles are circumscribed about two congruent triangles, the circles are equal.

Ex. **15**. If two circles are inscribed in two congruent triangles, the circles are equal.

Ex. **16**. To construct an isosceles triangle, given the base and the radius of the inscribed circle.

Ex. **17**. A circle is tangent internally at $P$ to another circle of twice the radius. Prove that the small circle bisects every chord of the larger circle drawn from $P$.

Ex. **18**. Four points, $A$, $B$, $C$, $D$, on a circle divide it into parts proportional to 6, 9, 10, 11. Find the number of degrees in all angles formed by joining the four points in succession, drawing the diagonals, and extending pairs of sides until they meet outside the circle.

# BOOK III

## PROPORTION. SIMILAR POLYGONS

**268.** DEF. A **proportion** is a statement of the equality of two ratios, as $\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$ (see p. 123).

NOTE. The statements $\frac{a}{b} = \frac{c}{d}$, and $a : b = c : d$ are equivalent. Hence, if a hypothesis states $a : b = c : d$, we may in the proof employ this statement in the form $\frac{a}{b} = \frac{c}{d}$ without assigning a special reason. Similarly, if we have to prove $a : b = c : d$, we may prove instead $\frac{a}{b} = \frac{c}{d}$.

**269.** DEF. The first and the fourth terms of a proportion are called the **extremes**; the second and the third, the **means.**

**270.** DEF. The first and the third terms are called the **antecedents**, the second and the fourth the **consequents**.

Thus, in the proportion, $a : b = c : d$, $a$ and $d$ are the extremes, $b$ and $c$ the means, $a$ and $c$ the antecedents, and $b$ and $d$ the consequents.

**271.** DEF. When the means of a proportion are equal, *either* mean is said to be the **mean proportional** between the first and the last terms.

Thus, in the proportion, $a : b = b : c$, $b$ is the mean proportional between $a$ and $c$.

**272.** DEF. The last term is the **fourth proportional** to the first three.

Thus, in the proportion, $a : b = c : d$, $d$ is the fourth proportional to $a$, $b$, and $c$.

**273.** The two terms of a ratio must be either quantities of the same denomination, or the quantities must be represented by their numerical measures only.

### Proposition I. Theorem

**274.** *In any proportion, the product of the means is equal to the product of the extremes.*

**Given** $a:b = c:d$.

**To prove** $ad = bc$.

**Proof.** $\frac{a}{b} = \frac{c}{d}$. (Hyp.)

Clearing of fractions, *i.e.* multiplying both members by $bd$,

$ad = bc$. (Ax. 7.)

**275.** Cor. *If any three terms of a proportion are respectively equal to the three corresponding terms of another proportion, the remaining terms are equal.*

**276.** Note. The product of two quantities, in geometry, means the product of the numerical measures of the quantities.

---

Ex. **1.** Determine the value of $x$, if

(*a*) $3 : x = 4 : 8$, (*c*) $x : 7 = 2 : 21$,

(*b*) $112 : 42 = 16 : x$, (*d*) $a : m = x : n$.

Ex. **2.** Find the fourth proportional to

(*a*) 1, 2, and 3, (*b*) 2, 1, and 3, (*c*) $m$, $n$, and $p$.

Ex. **3.** Find the mean proportional to

(*a*) 9 and 4, (*b*) 30 and 3, (*c*) 1 and 16.

### Proposition II. Theorem

**277.** *If the product of two numbers is equal to the product of two other numbers, either pair of numbers may be made the means, and the other pair the extremes, of a proportion.*

**Given** $ad = bc$.

**To prove** $a : b = c : d$.

**Proof.** $ad = bc$. (Hyp.)

Dividing both members by $bd$,

$$\frac{a}{b} = \frac{c}{d}. \qquad \text{(Ax. 8.)}$$

Ex. **1**. If $ab = mn$, find all possible proportions consisting of $a$, $b$, $m$, and $n$.

Ex. **2**. Form two proportions commencing with 3 from the equality $3 \times 10 = 5 \times 6$.

Ex. **3**. If $ab = xy$, form two proportions commencing with $b$.

Ex. **4**. Find the ratio of $x : y$, if

(*a*) $6x = 5y$,
(*b*) $9x = 2y$,
(*c*) $6x = y$,
(*d*) $mx = ny$,
(*e*) $(a + b)x = cy$,
(*f*) $mx + nx = py$,
(*g*) $4x - 6y = 0$,
(*h*) $ax - ay = bx - by$.

## Proposition III. Theorem

**278.** *A mean proportional between two quantities is equal to the square root of their product.*

**Given** $a : b = b : c$.

**To prove** $b = \sqrt{ac}$.

**Proof.** $a : b = b : c$. (Hyp.)

$b^2 = ac$. (274)

Extracting the square root of both members,

$$b = \sqrt{ac}.$$

Ex. Find the mean proportional between

(*a*) 2 and 18.
(*b*) $\frac{4}{3}$ and $\frac{25}{3}$.
(*c*) $2a$ and $32a$.
(*d*) $3b^2$ and $9a^2$.

### PROPOSITION IV. THEOREM

**279.** *If four quantities are in proportion, they are in proportion by alternation,* i.e. *the first term is to the third as the second is to the fourth.*

**Given** $a:b = c:d.$

**To prove** $a:c = b:d.$

**Proof.** $\frac{a}{b} = \frac{c}{d}.$ (Hyp.)

$ad = bc.$ (274)

$\therefore a:c = b:d.$ (277)

### PROPOSITION V. THEOREM

**280.** *If four quantities are in proportion, they are in proportion by inversion,* i.e. *the second term is to the first as the fourth is to the third*

**Given** $a:b = c:d.$

**To prove** $b:a = d:c.$

**Proof.** $a:b = c:d.$ (Hyp.)

$ad = bc.$ (274)

$\therefore b:a = d:c.$ (277)

---

Ex. Transform the proportion, $m:x = p:q$, so that $x$ becomes the fourth term; the first term; the third term.

### PROPOSITION VI. THEOREM

**281.** *If four quantities are in proportion, they are in proportion by composition,* i.e. *the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

**Given** $a:b = c:d.$

**To prove** $a+b:b = c+d:d.$

**Proof.** $a : b = c : d,$ (Hyp.)

or $$\frac{a}{b} = \frac{c}{d}.$$

$$\frac{a}{b} + 1 = \frac{c}{d} + 1. \qquad \text{(Why?)}$$

$$\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}. \qquad \text{(Why?)}$$

$$\therefore \frac{a+b}{b} = \frac{c+d}{d}, \qquad \text{(Why?)}$$

or $a + b : b = c + d : d.$

### PROPOSITION VII. THEOREM

**282.** *If four quantities are in proportion, they are in proportion by division,* i.e. *the difference between the first two terms is to the second term as the difference between the last two terms is to the fourth term.*

**Given** $a : b = c : d.$

**To prove** $a - b : b = c - d : d.$

**Proof.** $$\frac{a}{b} = \frac{c}{d}. \qquad \text{(Hyp.)}$$

$$\frac{a}{b} - 1 = \frac{c}{d} - 1, \qquad \text{(Ax. 3.)}$$

$$\therefore \frac{a-b}{b} = \frac{c-d}{d}. \qquad \text{(Sub.)}$$

---

Ex. **1.** Transform the following proportions so that only one term contains $x$.

(*a*) $2 : 3 = 5 - x : x.$
(*b*) $6 : 7 = 2 - x : x.$
(*c*) $a : b = 5 - x : x.$
(*d*) $4 : 3 = 2 + x : x.$
(*e*) $7 : 5 = 3 + x : x.$
(*f*) $a : b = 5 + x : x.$

Ex. **2.** If $x + y : y = 7 : 3$, find the ratio of $x$ and $y$.

Ex. **3.** If $x - y : y = 2 : 3$, find the ratio of $x$ and $y$.

### PROPOSITION VIII. THEOREM

**283.** *If four quantities are in proportion, they are in proportion by composition and division,* i.e. *the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

**Given** $a:b = c:d.$

**To prove** $a+b:a-b = c+d:c-d.$

**Proof.** $a:b = c:d.$ (Hyp.)

$$\frac{a+b}{b} = \frac{c+d}{d}. \quad (281)$$

$$\frac{a-b}{b} = \frac{c-d}{d}. \quad (282)$$

$$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}. \quad (\text{Ax. } 8.)$$

---

Ex. **1.** Transform the following proportions so that only one term contains $x$.

(*a*) $3:2 = 5+x:5-x.$
(*b*) $5:3 = 3+x:3-x.$
(*c*) $a:b = 1+x:1-x.$

Ex. **2.** If $x+y:x-y = 12:5$, find the ratio of $x$ to $y$.

Ex. **3.** If $x+y:x-y = a:b$, find the ratio of $x$ to $y$.

### PROPOSITION IX. THEOREM

**284.** *In a series of equal ratios, the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

**Given** $a:b = c:d = e:f.$

**To prove** $a+c+e:b+d+f = a:b.$

**Proof.** Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \cdots = r.$ (Hyp.)

Then $a = br,\ c = dr,\ e = fr,$ (Why?)

and $a + c + e = br + dr + fr$ (Ax. 2.)

or $a + c + e = (b + d + f)r$ (Sub.)

$\therefore \frac{a + c + e}{b + d + f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ (Why?)

---

Ex. **1.** If $a : b = c : d = e : f = 5 : 7$, find $\frac{a + c + e}{b + d + f}$.

Ex. **2.** If $\frac{x - a - c}{y - b - d} = \frac{a}{b} = \frac{c}{d}$, find $x : y$.

Ex. **3.** If $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, find the ratio $x + y + z$ to $z$.

### PROPOSITION X. THEOREM

**285.** *The products of the corresponding terms of two or more proportions are in proportion.*

**Given** $a : b = c : d.$

$m : n = p : q.$

**To prove** $am : bn = cp : dq$

**Proof.** $\frac{a}{b} = \frac{c}{d},\ \frac{m}{n} = \frac{p}{q}.$ (Hyp.)

$\therefore \frac{am}{bn} = \frac{cp}{dq}.$ (Ax. 7.)

**286.** COR. If $\frac{a}{b} = \frac{c}{d},$

then $\frac{ma}{nb} = \frac{mc}{nd}.$

---

Ex. **1.** If $x : y = 1 : 4$, and $x : \frac{1}{y} = 1 : 9$, find $x$.

Ex. **2.** If $\frac{1}{x} : y = 1 : 2$, and $x : y = 1 : 8$, find $y$.

### PROPOSITION XI. THEOREM

**287.** *If four quantities are in proportion, like powers or like roots of these quantities are in proportion.*

**Given** $a : b = c : d.$

**To prove** $a^n : b^n = c^n : d^n,$

and $\sqrt[m]{a} : \sqrt[m]{b} = \sqrt[m]{c} : \sqrt[m]{d}.$

**Proof.** $\frac{a}{b} = \frac{c}{d}.$ (Hyp.)

Raising both members to the $n$th power,

$\therefore \frac{a^n}{b^n} = \frac{c^n}{d^n}.$ (Why?)

Similarly, taking the $m$th root of each member,

$\therefore \frac{\sqrt[m]{a}}{\sqrt[m]{b}} = \frac{\sqrt[m]{c}}{\sqrt[m]{d}}.$ (Why?)

**288.** In elementary plane and solid geometry we need only squares and square roots, and cubes and cube roots. Prop. XI might read, for us, "If four quantities are in proportion, then the squares and square roots (also cubes and cube roots) of the quantities are in proportion."

---

Ex. **1.** If $x^3 : y^3 = 64 : 125$, find $\frac{x}{y}$.

Ex. **2.** If $\sqrt{x} : \sqrt{y} = 1 : 2$, find $x : y$.

Ex. **3.** If $\sqrt[3]{x} : \sqrt[3]{y} = 1 : 3$, find $x : y$.

Ex. **4.** If $\sqrt[3]{x} : 1 = \sqrt[3]{y} : 2$, find $\frac{x}{y}$.

Ex. **5.** If $x + y : x - y = 3 : 1$, find the ratio of $x^2$ to $y^2$.

Ex. **6.** If $x^2 : 4a^2 = y^2 : b^2$, find the ratio of $x^3$ to $y^3$.

### PROPOSITION XII. THEOREM

**289.** *Equimultiples of two quantities are in the same ratio as the quantities.*

**Given** $a$ and $b$ two quantities.

**To prove** $ma : mb = a : b$.

**Proof.** $a : b = a : b$. (Iden.)

$m : m = 1 : 1$. (Why?)

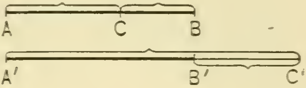
$\therefore ma : mb = a : b$. (Prop. X.)

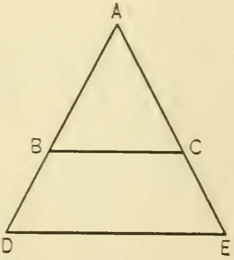
**290.** DEF. If in a line $AB$, or its prolongation, a point $C$ be taken, $AC$ and $BC$ are called **segments of the line.**

**291.** The segments are **internal** or **external** ones, according as $C$ lies in $AB$ or in the prolongation of $AB$.

Thus $AB$ is divided internally, $A'B'$ externally.

The segments of $AB$ are $AC$ and $BC$. The segments of $A'B'$ are $A'C'$ and $B'C'$.





NOTE. If $BC$ meets $AD$ and $AE$ so that $AB : BD = AC : CE$, the line $BC$ is said to divide the sides into *segments that are proportional.* If $AB : AD = AC : AE$, then the sides are said to be *divided proportionally.*

Ex. **1.** Divide a line 10 inches long internally in the ratio of $2 : 3$.

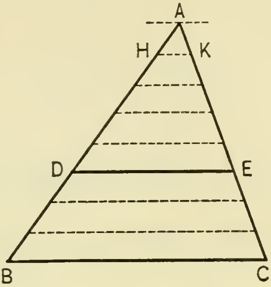
Ex. **2.** Divide a line 18 inches long externally in the ratio of $3 : 2$.

Ex. **3.** An 18-inch line is divided into segments in the continued ratio of $2 : 3 : 4$. Find the segments.

## PROPORTIONAL LINES

### Proposition XIII. Theorem[1]

**292.** *A line parallel to one side of a triangle divides the other two sides into segments that are proportional.*



Given in $\triangle ABC$, $DE$ parallel to $BC$.

To prove $AD : DB = AE : EC$.

### Proof

Case I. (Commensurable Case.)

| STATEMENTS | REASONS |
|---|---|
| 1. Let $AH$ be the unit of measure so that $AD = m(AH)$ and $DB = n(AH)$. | 1. By hyp., the two segments are commensurable. |
| 2. $\frac{AD}{DB} = \frac{m}{n}$. | 2. The ratio of two quantities of the same kind equals the ratio of their numerical measures. |
| 3. Through point $A$ and the various points of division, draw lines ∥ to $BC$. | 3. Why? |
| 4. These lines are ∥ to $BC$, $DE$, and to each other. | 4. Why? |

---

[1] For the proof based on principle of limits, see Appendix.

| | |
|---|---|
| **5.** $AE$ and $EC$ will be divided into $m$ and $n$ segments, each equal to $AK$; *i.e.* $AE = m(AK)$ and $EC = n\ (AK)$. | **5.** $\parallel$s cutting off equal distances on one transversal cut off equal distances on every transversal. |
| **6.** $\frac{AE}{EC} = \frac{m}{n}$. | **6.** Why? |
| **7.** $\therefore AD : DB = AE : EC$. | **7.** Why? |

[Practical applications, pages 291–296]

CASE II. $\frac{AD}{DB}$ is an irrational number.

Let $\frac{AD}{DB} = \sqrt{3} = 1.732 \cdots$

If $\frac{AD}{DB} = 1.7$, then $\frac{AE}{EC} = 1.7$. (Case I.)

If $\frac{AD}{DB} = 1.73$, then $\frac{AE}{EC} = 1.73$, and so forth. (Case I.)

Hence *all* approximate values of $\frac{AD}{DB}$ and $\frac{AE}{EC}$ are respectively equal.

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \qquad (223)$$

**293.** REMARK. If the transversal intersects two sides of the triangle, these sides are divided internally in the same ratio; if it meets their prolongations, the sides are divided externally in the same ratio.

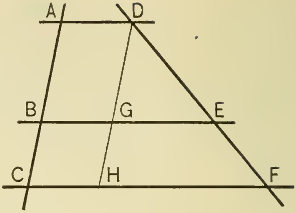
**294.** COR. I. *If a line parallel to one side of a triangle intersects the other two sides, either side is to one of its segments as the other side is to its corresponding segment.*

**Proof:** Using figure for the proposition, we can show that $AB = (m + n)(AH)$ and $AD = m(AH)$.

Hence, $\frac{AB}{AD} = \frac{m + n}{m}$.

Likewise, $\frac{AC}{AE} = \frac{m + n}{m}$ and $\therefore \frac{AB}{AD} = \frac{AC}{AE}$.

NOTE. This corollary is of importance in proving relations in similar triangles.



**295.** Cor. 2. *Three or more parallel lines cut off proportional segments on any two transversals.*

Hint. Draw $DH \parallel AC$, $AB = DG$, $BC = GH$.

Ex. **1.** In the diagram for Prop. XIII, if $AD = 4$, $DB = 8$, $AE = 3$, find $EC$.

Ex. **2.** In the same diagram, find $DB$, if $AD = a$, $AE = b$, and $EC = c$.

Ex. **3.** In the same diagram, find $AE$, if $AB = 12$, $AD = 8$, and $AC = 9$.

Ex. **4.** In the same diagram, find $AE$, if $AB = m$, $AD = n$, and $AC = p$.

Ex. **5.** In the same diagram, find $AD$, if $AD = EC$, $DB = 4$, and $AE = 9$.

Ex. **6.** In the same diagram, find $AE$, if $AE = 2\,DB$, $AD = 10$, and $EC = 20$.

Ex. **7.** In the same diagram, find $EC$ if $EC = AD$, $AE = m$, and $BD = n$.

Ex. **8.** In the same diagram, find $EC$, if $AB = a$, $AD = b$, and $AC = c$.

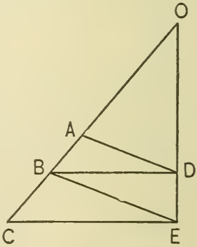
Ex. **9.** In the same diagram if $DE \parallel BC$, and $AD : DB = EC : AE$, then $AE = EC$.

Ex. **10.** In the same diagram, if $AD = 2(AE)$ and $DB = 6$, find $EC$.

Ex. **11.** If in the diagram for Prop. XIII, $AD = 2$, $DB = 3$, $AE = 4$, and $EC = 4$, is $DE$ parallel to $BC$?

Ex. **12.** In the diagram, if $BD \parallel CE$, and $AD \parallel BE$, then

$$OA : OB = OB : OC.$$



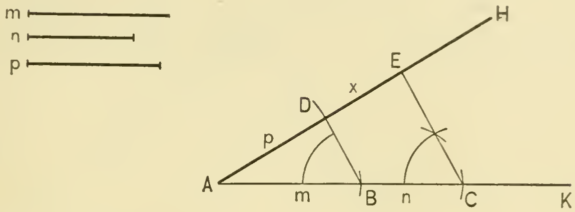
Ex. 12

Ex. **13.** In diagram for Prop. XIII, find $DB$ and $EC$, if $AD = 8$, $DB = AE$, and $AC = 6$.

Ex. **14.** In diagram for Prop. XIII, find $DB$ and $AE$, if $AD = 8$, $AE =$ one half $DB$, and $EC = 1$.

### PROPOSITION XIV. PROBLEM

**296.** *To find the fourth proportional to three given lines.*



**Given** three lines $m$, $n$, and $p$.

**Required** the fourth proportional to $m$, $n$, and $p$.

**Construction.** Draw any angle $KAH$.

On $AK$, make $AB = m$, $BC = n$; on $AH$, make $AD = p$.

Draw $BD$.

Through $C$, draw a line parallel to $BD$, meeting $AH$ in $E$. $DE$ is the required fourth proportional.

[The proof is left to the student. HINT. Apply Prop. XIII.]

---

Ex. **1**. If $a$, $b$, and $c$ are given lines, construct a line $x$, so that

$$a : b = x : c.$$

Ex. **2**. If $a$, $b$, and $c$ are given lines, construct a line equal to $\frac{bc}{a}$.

Ex. **3**. If $a$, $b$, and $c$ are given lines, construct a line equal to $\frac{bc}{2a}$.

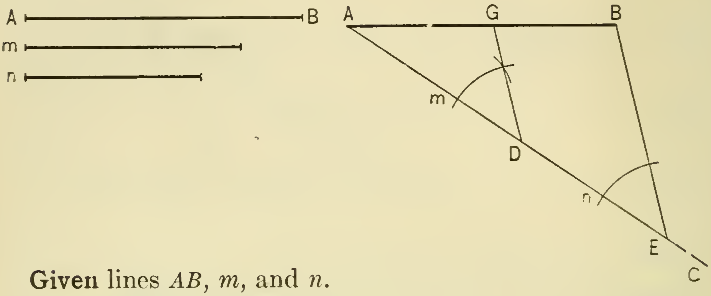
Ex. **4**. If $a$ and $b$ are two given lines, construct a line $x = \frac{b^2}{a}$; to $\frac{2b^2}{3a}$.

Ex. **5**. Construct $x = a(a + b) \div b$.

Ex. **6**. Construct the fourth proportional to three lines $\frac{7}{8}''$, $\frac{3}{4}''$, and $1''$ long.

## Proposition XV. Problem

**297.** *To divide a given line into segments proportional to two given lines.*



**Given** lines $AB$, $m$, and $n$.

**Required** to divide $AB$ into segments proportional to $m$ and $n$.

**Construction.** At any angle with $AB$ draw $AC$.

On $AC$, cut off $AD = m$, $DE = n$.

Draw $EB$.

Through $D$ draw a parallel to $BE$, intersecting $AB$ in $G$.

Then $AB$ is divided as required.

[The proof is left to the student. Hint. Apply Prop. XIII.]

Ex. **1**. Divide a given line $AC$ into two segments having the ratio of 2 : 3.

Ex. **2**. Divide a 2″ line into two segments having the ratio that a diagonal of a square has to one of its sides.

Ex. **3**. Divide a given line into 3 segments having the continued ratio of three lines $s$, $t$, and $w$.

Ex. **4**. Construct two lines when their sum and their ratio are given.

Ex. **5**. Construct two lines when their difference and their ratio are given.

Ex. **6**. In a given line, $AB$, to find a point, $C$, so that

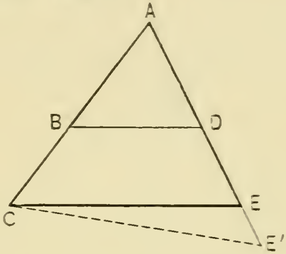
$$AB : AC = m : n,$$

when $m$ and $n$ are two given lines.

### PROPOSITION XVI. THEOREM

**298.** *If a line divides two sides of a triangle into segments that are proportional, it is parallel to the third side.*

[Converse to Prop. XIII.]



**Given** in $\triangle AEC$, $AB : BC = AD : DE$.

**To prove** $DB$ parallel to $EC$.

**Proof.** **1.** Through $C$, draw $CE' \parallel BD$, meeting $AE$ in $E'$. (Why?)

**2.** $AB : BC = AD : DE$. (Hyp.)

**3.** $AB : BC = AD : DE'$. (292)

**4.** $\therefore DE = DE'$. (275)

**5.** Point $E$ coincides with point $E'$. (Why?)

**6.** $\therefore CE$ and $CE'$ coincide. (Why?)

**7.** $\therefore BD \parallel CE$. (Why?)

**299.** COR. *If a line divides two sides of a triangle so that one side is to one of its segments as the second side is to its corresponding segment, then the line is parallel to the third side.*

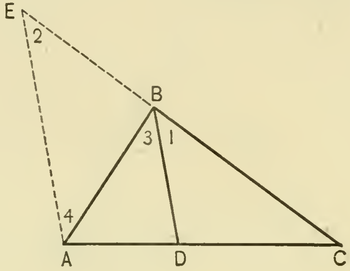
[Converse to Prop. XIII. Cor. 1.]

**300.** REMARK. This corollary is often stated: "If a line divides two sides of a triangle proportionally, it is parallel to the third side."

EX. In the diagram for Prop. XVI, if $AB = 12$, $BC = 16$, $AD = 15$ $DE = 20$, is $BD \parallel CE$?

## Proposition XVII. Theorem

**301.** *The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*



**Given** in $\triangle ABC$, $BD$ bisecting $\angle ABC$.

**To prove** $AB : BC = AD : DC$.

**Proof.** 1. Draw $AE \parallel DB$, to meet $CB$, produced, in $E$. (Why?)

2. $\angle 1 = \angle 2$. (106)

3. $\angle 3 = \angle 4$. (104)

4. $\angle 1 = \angle 3$. (Hyp.)

5. $\angle 2 = \angle 4$. (Ax. 1.)

6. $AB = BE$. (121)

7. $EB : BC = AD : DC$. (292)

8. $\therefore AB : BC = AD : DC$. (Sub.)

**302.** Theorem. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*

Note. Show that this theorem has no meaning if the bisector of the exterior angle is at the vertex of an isosceles triangle.

Ex. 1. In the diagram for Prop. XVII, find $DC$, if $AB = 3$, $BC = 4$, and $AD = 2$.

Ex. 2. In the same diagram, find $AD$, if $AB = m$, $BC = n$, and $DC = p$.

Ex. 3. In the same diagram, find $DC$, if

(*a*) $AB = 4, \quad BC = 5, \quad AC = 6.$
(*b*) $AB = 18, \; BC = 9, \quad AC = 21.$
(*c*) $AB = 21, \; BC = 14, \; AC = 25.$

Ex. 4. In the same diagram, find $AD$ and $DC$, if $BC = a$, $CA = b$, and $AB = c$.

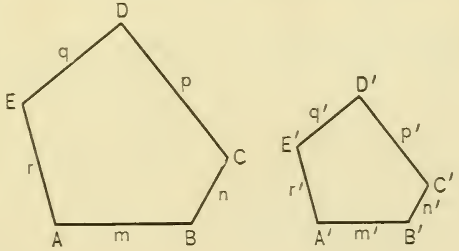
Ex. 5. In the same diagram, find $AB$, if $AB = DC$, $AD = 4$, and $BC = 16$.

Ex. 6. *To divide one side of a triangle into segments proportional to the other two sides.*

Ex. 7. State and prove the converse of Prop. XVII.

## SIMILAR POLYGONS

**303.** Def. Two polygons are **similar** if their corresponding angles are equal and their corresponding sides are proportional.



Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar, if

(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

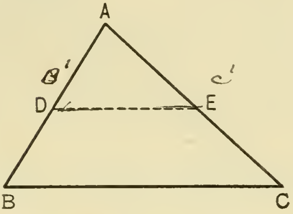
(2) $\frac{m}{m'} = \frac{n}{n'} = \frac{p}{p'}$, etc.

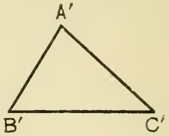
---

Ex. In two similar rectangles the bases are 12″ and 8″ respectively. If the altitude of the first is 9″ what is the altitude of the second rectangle?

## PROPOSITION XVIII. THEOREM

**304.** *Two triangles are similar if the three angles of one are equal respectively to the three angles of the other.*





**Given** in $\triangle ABC$ and $A'B'C'$,

$$\angle A = \angle A', \ \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

**To prove** $\triangle ABC \sim \triangle A'B'C'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Place $\triangle A'B'C'$ upon $\triangle ABC$, $\angle A'$ on $\angle A$, taking the position $\triangle ADE$. | 1. Why? |
| 2. $\angle ADE = \angle B' = \angle B$. | 2. Hyp. |
| 3. $DE \parallel BC$. | 3. If 2 cor. $\angle$s are $=$, lines are $\parallel$. |
| 4. $AB : AD = AC : AE$. | 4. (294) |
| 5. $AB : A'B' = AC : A'C'$. | 5. Sub. |
| 6. By placing $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle B'$ coincides with $\angle B$, we can show that $AB : A'B' = BC : B'C'$. | 6. Steps similar to (1–5). |
| 7. $\therefore \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$. | 7. Why? |
| 8. But $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$. | 8. Hyp. |
| 9. $\therefore \triangle ABC \sim \triangle A'B'C'$. | 9. Their corr. $\angle$s are $=$.<br>Their corr. sides are proportional. |

**305**. Cor. 1. *Two triangles are similar if two angles of the one are equal respectively to two angles of the other.*

**306**. Cor. 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

**307**. Cor. 3. *A line parallel to one side of a triangle cuts off a triangle similar to the given triangle.*

**308**. Cor. 4. *If two triangles are similar to a third triangle, they are similar to each other.*

---

Ex. **1**. The sides of a polygon are 3, 4, 5, 6, and 7. Find the sides of a similar polygon if the side corresponding with 3 equals 15.

Ex. **2**. The sides of a triangle are $a$, $b$, and $c$. Find the sides of a similar triangle if the side corresponding to $a$ is equal to $m$.

Ex. **3**. If two chords $AB$ and $CD$ intersect in $E$, the triangle $AEC$ is similar to the triangle $BED$.

Ex. **4**. If from a point $A$ without a circle, two secants are drawn to meet the circle in $B$ and $C$, and $D$ and $E$, respectively, the triangle $ABE$ is similar to the triangle $ACD$.

Ex. **5**. If the altitudes $AD$ and $BE$ of the triangle $ABC$ intersect in $F$, the triangle $AFE$ is similar to the triangle $BFD$.

Ex. **6**. Prove: Equilateral triangles are similar.

Ex. **7**. Prove: *Two isosceles triangles having equal vertex (or equal base) angles are similar.*

Ex. **8**. Show two squares are always similar

Ex. **9**. *The diagonals of a trapezoid divide each other into segments that are proportional.*

Ex. **10**. If $AD$, the angle bisector of the inscribed triangle $ABC$, is produced to meet the circle in $E$, the triangle $ABD$ is similar to the triangle $AEC$.

Ex. **11**. Two circles are tangent internally. Prove that all chords of the larger circle drawn from the point of contact are divided in the same ratio by the smaller circle.

**309.** Method XVII. **Similar triangles are the usual means of proving that lines are proportional.** To prove, therefore, that four lines are proportional:

(1) *Select two triangles so that each contains two of the given lines.* (It is advisable to mark the lines as indicated in 254 and 255.)

(2) *Prove the similarity of the two triangles.* (If triangles are not similar, select another pair.)

(3) *Derive the proportion.*

(4) (*Apply alternation and inversion*, if necessary.)

---

Ex. **1.** If in the triangle $ABC$ the altitudes $AD$ and $BE$ be drawn, prove that $AC : BC = DC : EC$.

Ex. **2.** In the same diagram, if $AD$ and $BE$ meet in $F$, prove $BF : FA = DF : FE$.

Ex. **3.** In the same diagram, $AE : AD = FE : DC$.

Ex. **4.** If from the vertex $A$ of an inscribed triangle $ABC$ the altitude $AD$ and the diameter $AF$ be drawn, then $AB : AD = AF : AC$.

Ex. **5.** In the same diagram, $BD : FC = AD : AC$.

Ex. **6.** *If from a point without a circle a tangent and a secant be drawn, the tangent is the mean proportional between the secant and its external segment.*

Ex. **7.** If a diameter $AB$ be produced to $C$, at $C$ a perpendicular be erected, and through $B$ a line be drawn to meet the circle and the perpendicular in $D$ and $E$ respectively, then $AB : BE = DB : BC$.

Ex. **8.** If in a right triangle $ABC$ the altitude $AD$ be drawn upon the hypotenuse, $AD : AB = AC : BC$.

Ex. **9.** In the same diagram, $AD : AB = DC : AC$.

Ex. **10.** If the angle-bisector $CD$ of an inscribed triangle $ABC$ be produced to meet the circle in $E$, $EB : EC = DB : CB$.

Ex. **11.** If $CD$, the angle-bisector of the inscribed triangle $ABC$, be produced to meet the circle in $E$, then $EB$ is a mean proportional between $CE$ and $DE$.

Ex. **12.** In the same diagram, $AD : EB = AC : CE$.

Ex. **13.** *In similar triangles corresponding angle bisectors have the same ratio as any two corresponding sides.*

Ex. **14.** *In similar triangles corresponding altitudes have the same ratio as any two corresponding sides.*

**310.** METHOD XVIII. **To prove that the product of two lines equals the product of two other lines**, *use the method of* (309) *and take the products of the means and extremes of the resulting proportion.*

---

Ex. **1.** *If two chords intersect within a circle, the product of the segments of one equals the product of the segments of the other.*

Ex. **2.** If from any point $E$ in the chord $AB$ the perpendicular $EC$ be drawn upon the diameter $AD$, then $AC \times AD = AB \times AE$.

Ex. **3.** *The product of two arms of a right triangle is equal to the product of the hypotenuse and the altitude upon the hypotenuse.*

Ex. **4.** If in triangle $ABC$, a parallel to $AB$ meets $BC$ and $CA$ in $E$ and $D$ respectively, then $AC \times DE = DC \times AB$.

Ex. **5.** The product of any altitude of a triangle and its corresponding side is equal to the product of any other altitude and its corresponding side.

Ex. **6.** The three sides of triangles are 14, 15, 13, and the altitude upon 14 equals 12. Find the other altitudes. (Compare Ex. 5.)

Ex. **7.** If in triangle $ABC$ the altitudes $AD$ and $BE$ meet in $F$, then $BF \times BE = BC \times BD$.

Ex. **8.** If in the triangle $ABC$ the altitudes $AD$ and $BE$ meet in $F$, then $BD \times DC = DF \times AD$.

Ex. **9.** If $AB$ is a diameter, $BD$ the tangent at $B$, and $DA$ meets the circle in $E$, then $\overline{AB}^2 = AE \times AD$.

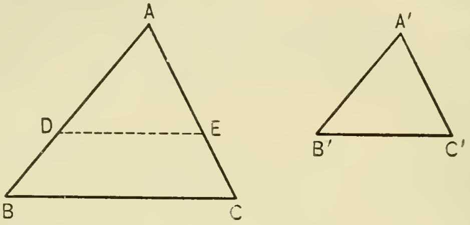
Ex. **10.** In the same diagram, $\overline{BE}^2 = AE \times ED$.

Ex. **11.** If the sides $AB$ and $DC$ of inscribed quadrilateral $ABCD$ are produced until they meet in $E$, and $\angle DBA = \angle CBE$, then $AD \times BE = CE \times BD$.

Ex. **12.** If in $\triangle ABC$ the altitudes $AD$ and $BE$ are drawn and $BE = 6$, $EC = 3$, $DC = 2$, find $AD$.

## Proposition XIX. Theorem

**311.** *Two triangles are similar if an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*



**Given** in $\triangle s\ ABC$ and $A'B'C'$,

$$\angle A = \angle A', \text{ and } AB : A'B' = AC : A'C'.$$

**To prove** $\triangle ABC \sim \triangle A'B'C'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$ and $\triangle ABC$ takes position $\triangle DAE$. | **1.** Why? |
| **2.** $AB : A'B' = AC : A'C'$. | **2.** Hyp. |
| **3.** $AB : AD = AC : AE$. | **3.** Sub. |
| **4.** $DE \parallel BC$. | **4.** Prop. XVI, cor. |
| **5.** $\angle B = \angle ADE$ and $\angle C = \angle DEA$. | **5.** Why? |
| **6.** $\therefore \triangle ABC \sim \triangle ADE$ (or $\triangle A'B'C'$). | **6.** Prop. XVIII. |

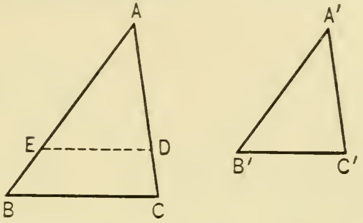
Ex. **1.** *In similar triangles, corresponding medians have the same ratio as any two corresponding sides.*

Ex. **2.** Two triangles, $ABC$ and $A'B'C'$, are similar if altitude $AD$ : altitude $A'D' = BC : B'C'$, and $\angle B = \angle B'$.

Ex. **3.** In similar triangles, the radii of the circumscribed circles have the same ratio as any two corresponding sides.

### PROPOSITION XX. THEOREM

**312.** *Two triangles are similar if their corresponding sides are proportional.*



**Given** in $\triangle ABC$ and $A'B'C'$, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

**To prove** $\triangle ABC \sim \triangle A'B'C'$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. On $AC$ and $AB$ respectively, lay off $AD = A'C'$ and $AE = A'B'$. | 1. Why? |
| 2. Draw $DE$. | 2. Why? |
| 3. $AB : A'B' = AC : A'C'$. | 3. Hyp. |
| 4. $AB : AE = AC : AD$. | 4. Sub. |
| 5. $\triangle ADE \sim \triangle ABC$. | 5. (311). |
| 6. $AB : AE = BC : ED$. | 6. (303). |
| 7. But $AB : A'B' = BC : B'C'$. | 7. Hyp. |
| 8. $ED = B'C'$. | 8. (275). |
| 9. $\therefore \triangle A'B'C' \cong \triangle ADE$. | 9. (s. s. s. = s. s. s.). |
| 10. $\therefore \triangle A'B'C' \sim \triangle ABC$. | 10. Sub. |

---

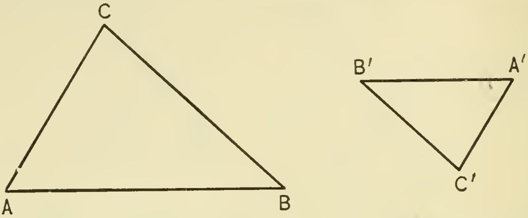
Ex. 1. The lines which join the mid-points of the sides of $\triangle ABC$ form a triangle similar to $\triangle ABC$.

Ex. 2. *Two triangles are similar if two sides and the median to one of these sides are proportional to the corresponding parts of the other.*

* Ex. 3. Two right triangles are similar if the hypotenuse and an arm of one triangle are proportional to the hypotenuse and an arm of the other.

## PROPOSITION XXI. THEOREM

**313.** *Two triangles are similar if the sides of one are respectively parallel to the sides of the other.*



**Given** in $\triangle$s $ABC$ and $A'B'C'$, $AB \parallel A'B'$, $AC \parallel A'C'$, and $BC \parallel B'C'$

**To prove** $\triangle ABC \sim \triangle A'B'C'$.

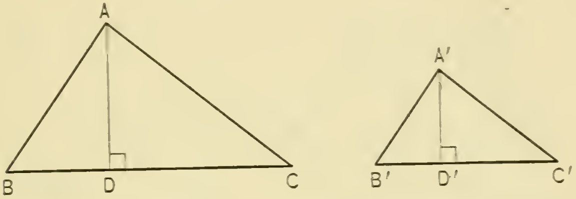
### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Either $\begin{cases} \angle A + \angle A' = 2 \text{ rt. } \angle\text{s, or } \angle A = \angle A'; \\ \angle B + \angle B' = 2 \text{ rt. } \angle\text{s, or } \angle B = \angle B'; \\ \angle C + \angle C' = 2 \text{ rt. } \angle\text{s, or } \angle C = \angle C'. \end{cases}$ | **1.** Why? |
| **2.** If $\angle A + \angle A' = 2$ rt. $\angle$s, $\angle B + \angle B' = 2$ rt. $\angle$s and $\angle C + \angle C' = 2$ rt. $\angle$s, then $\angle A + \angle B + \angle C + \angle A' + \angle B' + \angle C' = 540°$. | **2.** Why? |
| **3.** This is impossible. | **3.** Sum of $\angle$s of 2 $\triangle$s $= 360°$. |
| **4.** If $\angle A + \angle A' = 2$ rt. $\angle$s, $\angle B + \angle B' = 2$ rt. $\angle$s, and $\angle C = \angle C'$, then $\angle A + \angle B + \angle C + \angle A' + \angle B' + \angle C' = 360° + 2\angle C$. | **4.** Why? |
| **5.** This is impossible. | **5.** Same as 3. |
| **6.** $\therefore \angle C = \angle C'$ and, say, $\angle B = \angle B'$. | **6.** Why? |
| **7.** $\therefore \triangle ABC \sim \triangle A'B'C'$. | **7.** (305) |

EX. *Two triangles are similar if the sides of one are respectively perpendicular to the sides of the other.* HINT. Follow Prop. XXI.

## PROPOSITION XXII. THEOREM

**314.** *If two triangles are similar, corresponding altitudes have the same ratio as any two corresponding sides.*



Given $AD$ and $A'D'$, the corresponding altitudes of the similar triangles $ABC$ and $A'B'C'$.

To prove $\quad \dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. $\triangle ABC \sim \triangle A'B'C'$. | 1. Hyp. |
| 2. $\angle BDA = \angle B'D'A'$. | 2. Why? |
| 3. $\angle B = \angle B'$. | 3. Why? |
| 4. Rt. $\triangle ABD \sim$ rt. $\triangle A'B'D'$. | 4. (306). |
| 5. $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'}$. | 5. (303). |
| 6. But $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$. | 6. Why? |
| 7. $\therefore \dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$. | 7. Why? |

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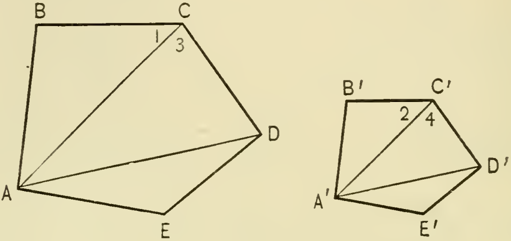
Ex. 1. The base of a triangle is 2 ft. and the altitude 9 in. If the corresponding base of a similar triangle is 6 in., find the corresponding altitude.

Ex. 2. Prove: In two similar triangles corresponding altitudes divide the triangles into similar triangles.

### Proposition XXIII. Theorem

**315.** *Two similar polygons may be divided into the same number of similar triangles, similar each to each, and similarly placed.*



**Given** polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

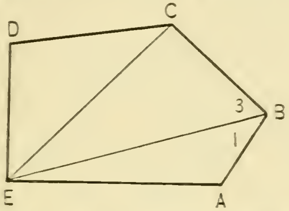
**To prove** $\triangle ABC \sim \triangle A'B'C'$,
$\triangle ACD \sim \triangle A'C'D'$, etc.

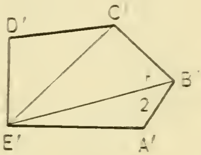
### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. $AB : A'B' = BC : B'C' = CD : C'D'$, etc. $\angle B = \angle B', \angle C = \angle C', \angle D = \angle D'$. | 1. Why? |
| 2. $\triangle ABC \sim \triangle A'B'C'$. | 2. (311). |
| 3. $\angle 1 = \angle 2$. | 3. Why? |
| 4. $\angle 3 = \angle 4$. | 4. Why? |
| 5. $\frac{BC}{B'C'} = \frac{CD}{C'D'}$. | 5. Hyp. |
| 6. $\frac{BC}{B'C'} = \frac{CA}{C'A'}$. | 6. (303). |
| 7. $\frac{CA}{C'A'} = \frac{CD}{C'D'}$. | 7. Why? |
| 8. $\triangle ACD \sim \triangle A'C'D'$. | 8. (311). |
| 9. Similarly we can prove $\triangle ADE \sim \triangle A'D'E'$, etc. | 9. Steps (1–8). |

## PROPOSITION XXIV. THEOREM

**316.** *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*





**Given** in the polygons $ABCDE$ and $A'B'C'D'E'$, $\triangle ABE \sim \triangle A'B'E'$, $\triangle BCE \sim \triangle B'C'E'$, $\triangle CDE \sim \triangle C'D'E'$, etc., and similarly placed.

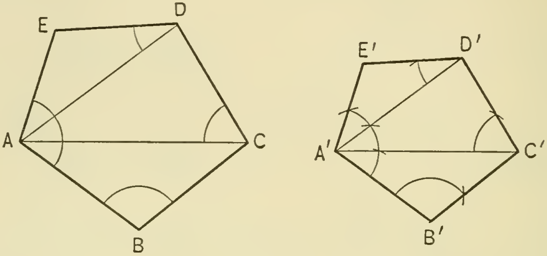
**To prove** polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. $\triangle ABE \sim \triangle A'B'E'$. | 1. Hyp. |
| 2. $\frac{AB}{A'B'} = \frac{BE}{B'E'}$, $\angle 1 = \angle 2$. | 2. Def. $\sim$ $\triangle$. |
| 3. $\triangle EBC \sim \triangle E'B'C'$. | 3. Hyp. |
| 4. $\frac{BE}{B'E'} = \frac{BC}{B'C'}$, $\angle 3 = \angle 4$. | 4. Same as 2. |
| 5. $\frac{AB}{A'B'} = \frac{BC}{B'C'}$. | 5. Why? |
| 6. $\angle 1 + \angle 3 = \angle 2 + \angle 4$. | 6. Why? |
| 7. $\angle B = \angle B'$. | 7. Sub. |
| 8. Likewise,<br>$\frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.,<br>and $\angle C = \angle C'$, $\angle D = \angle D'$, etc. | 8. Steps (1–7). |
| 9. $\therefore$ Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$. | 9. (303). |

### PROPOSITION XXV. PROBLEM

**317.** *To construct a polygon similar to a given polygon upon a line corresponding to a side of the given polygon.*



**Given** polygon $ABCDE$, and side $A'B'$ corres. to side $AB$.

**Required** to construct a polygon similar to $ABCDE$.

**Construction.** Draw all diagonals from one vertex, as $A$, in the given polygon. With $A'B'$ as a side and $A'$ and $B'$ as vertices construct $\angle B'A'C' = \angle BAC$ and $\angle B' = \angle B$. Extend sides to meet at $C'$. On $A'C'$ at $A'$ and $C'$ construct $\angle C'A'D' = \angle CAD$ and $\angle A'C'D' = \angle ACD$, etc.

**Proof:** $\triangle A'B'C' \sim \triangle ABC$, $\triangle C'A'D' \sim \triangle CAD$, etc. (Why?) Hence polygon $A'B'C'D'E' \sim ABCDE$. (Why?)

---

Ex. **1.** Draw polygons of 4, 5, and 6 sides. On a given line corresponding to a side in each of the polygons, construct polygons of 4, 5, and 6 sides similar to the given polygons.

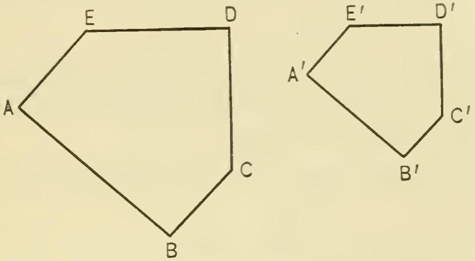
Ex. **2.** Draw a parallelogram. On $AB$ as a base construct a similar parallelogram.

Ex. **3.** To construct a quadrilateral $ABCD$ similar to a given quadrilateral $A'B'C'D'$ and having the diagonal $AC$ equal a given line.

Ex. **4.** Using Prop. XXV, show how to construct one rectangle similar to another rectangle.

## PROPOSITION XXVI. THEOREM

**318.** *The perimeters of two similar polygons are to each other as any two corresponding sides.*



**Given** $P$ and $P'$, the perimeters of the similar polygons $ABCDE$ and $A'B'C'D'E'$ respectively.

**To prove** $P : P' = AB : A'B' = BC : B'C' = CD : C'D'$, etc.

### Proof

| | STATEMENTS | REASONS |
|---|---|---|
| 1. | $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \cdots$ | 1. Hyp. |
| 2. | $\frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'} = \frac{BC}{B'C'} \cdots.$ | 2. (284). |
| 3. | $\therefore \quad \frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \cdots.$ | 3. Sub. |

---

Ex. **1.** *The perimeters of two similar polygons are to each other as any two corresponding diagonals.*

Ex. **2.** The perimeters of two similar triangles are to each other as any two corresponding altitudes.

Ex. **3.** The sides of a polygon are 4, 5, 6, 7, 8 respectively. Find the perimeter of a similar polygon, if the side corresponding to 5 is 7.

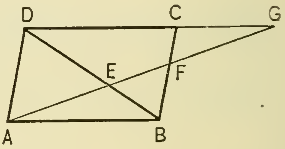
Ex. 4. The perimeters of two similar polygons are 20 and 25 inches, respectively. If a side of the first polygon is 4 in., find the corresponding side of the second polygon.

Ex. 5. In the diagram for Prop. XXVI find the perimeter of $ABCDE$, if the perimeter of $A'B'C'D'E'$ equals 20 inches, $A'B' = 4$ inches, $B'C' = 3$ inches, $AC = 10$ inches, $B'C' : A'B' = A'B' : A'C'$ and $ABCDE \sim A'B'C'D'E'$.

**319.** METHOD XIX. **To prove the proportionality of four lines which do not form similar triangles, find a third ratio equal to each of the given ones.**

Thus, in the annexed figure, if $ABCD$ is a parallelogram and $AG$ is a straight line, then

$$\frac{EF}{EA} = \frac{EA}{EG}.$$



Obviously no two triangles exist, each of which contains two of the four lines. Hence we have to find a third ratio equal to each of the given ones. This ratio is $\frac{EB}{ED}$. Our problem is therefore split into two, viz.:

(*a*) To prove $\frac{EF}{EA} = \frac{EB}{ED}$, and

(*b*) To prove $\frac{EA}{EG} = \frac{EB}{ED}$.

Each of these proportions is easily proved by means of the fundamental method (XVII).

---

Ex. 1. *In similar triangles the radii of the inscribed circles have the same ratio as any two corresponding sides.*

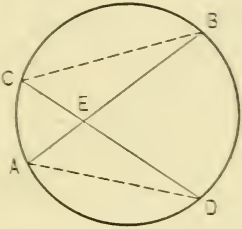
Ex. 2. If triangle $ABC$ is similar to triangle $A'B'C'$, and $AD$ and $A'D'$ are angle-bisectors, $AF$ and $A'F'$ are altitudes, prove that $AD : A'D' = AF : A'F'$

Ex. 3. If in the similar triangles $ABC$ and $A'B'C'$, the points $D$ and $D'$ are taken respectively in $BC$ and $B'C'$, so that $\angle BAD = \angle B'A'D'$, then $BD : B'D' = BC : B'C'$.

Ex. 4. Prove that the $\triangle$ $ADF$ and $A'D'F'$, in Ex. 2, are similar.

## PROPOSITION XXVII. THEOREM

**320.** *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



**Given** in $\odot O$, the chords $AB$ and $CD$ intersecting in $E$.

**To prove** $AE \times EB = CE \times ED$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $CB$ and $AD$. | 1. Why? |
| **2.** $\angle A = \angle C$, $\angle B = \angle D$. | 2. Meas. by $\frac{1}{2}\overset{\frown}{BD}$ and $\frac{1}{2}\overset{\frown}{AC}$ resp. |
| **3.** $\triangle CEB \sim \triangle AED$. | 3. Why? |
| **4.** $AE : CE = ED : EB$. | 4. (303). |
| **5.** $\therefore AE \times EB = CE \times ED$. | 5. (274). |

**321.** COR. *The product of the segments of a chord that passes through a fixed point within a circle is the same for all directions of the chord.*

---

Ex. **1**. In the diagram for Prop. XXVII, if $AE = 3$, $EB = 4$, $ED = 6$, find $CE$.

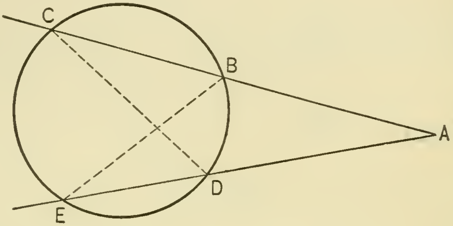
Ex. **2**. In the same diagram, if $AE = a$, $EB = b$, and $ED = c$, find $CE$.

Ex. **3**. In the same diagram, if $AE = 4$, $EB = 9$, and $CE = ED$, find $CE$.

Ex. **4**. If the prolongations of two chords meet without a circle, is Prop. XXVII correct for the external segments of the chords?

### PROPOSITION XXVIII. THEOREM

**322.** *If from a point without a circle, two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*



**Given** two secants $AC$ and $AE$ cutting a circle in $B$, $C$, $D$, and $E$ respectively.

**To prove** $AC \times AB = AE \times AD$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $CD$ and $EB$. | **1.** Why? |
| **2.** $\angle C = \angle E$. | **2.** (232). |
| **3.** $\angle A = \angle A$. | **3.** Iden. |
| **4.** $\angle CDA = \angle EBA$. | **4.** Why? |
| **5.** $\triangle CDA \sim \triangle EBA$. | **5.** (304). |
| **6.** $AC : AE = AD : AB$. | **6.** (303). |
| **7.** $\therefore AC \times AB = AE \times AD$. | **7.** (274). |

---

Ex. **1.** If, in the diagram for Prop. XXVIII, $CB = 16$, $BA = 2$, $DA = 4$, find $DE$.

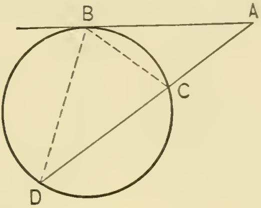
Ex. **2.** In the same diagram, if $CA : DA = 2 : 1$, prove that $EA = 2\,BA$.

Ex. **3.** If in the same diagram $CB = a$, $BA = b$, $AE = c$, find $DA$.

Ex. **4.** In the same diagram, if $AB = AD$, then $BCED$ is an isosceles trapezoid.

## Proposition XXIX. Theorem

**323.** *If from a point without a circle, a tangent and a secant are drawn, the tangent is the mean proportional between the secant and its external segment.*



**Given** the tangent $AB$ touching the $\odot BDC$ in $B$, and the secant $AD$ cutting the circle in $C$ and $D$.

**To prove** $AD : AB = AB : AC$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $BD$ and $BC$. | 1. Why? |
| 2. $\angle D = \angle CBA$. | 2. Why? |
| 3. $\angle A = \angle A$. | 3. Iden. |
| 4. $\angle DBA = \angle BCA$. | 4. Why? |
| 5. $\triangle DBA \sim \triangle BCA$. | 5. (304). |
| 6 $\therefore AD : AB = AB : AC$. | 6. (303). |

**324.** *If from a given point outside a circle, a secant is drawn, then the product of the whole secant times its external segment is constant, for all directions of the secant.*

---

Ex. **1.** If in the diagram for Prop. XXIX, $DC = 5$, $AC = 4$, find $AB$.

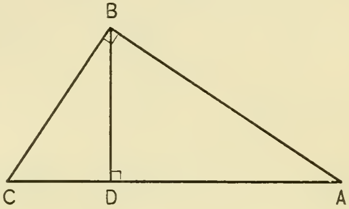
Ex. **2.** In the same diagram, if $AB = 8$ inches, and the distance of $A$ from the center of the circle equals 17 inches, find the radius of the circle.

Ex. **3.** Tangents to two intersecting circles drawn from any point in the common chord, produced, are equal.

Ex. **4.** If two circles are tangent externally and from any point in the common internal tangent secants are drawn to the two circles, the products of the secants and their external segments will be equal.

## PROPOSITION XXX. THEOREM

**325.** *In any right triangle, the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles, similar to each other and to the given triangle.*



**Given** in the rt. $\triangle ACB$, $BD$ the $\perp$ to hy. $CA$.

**To prove** $\triangle BCD \sim \triangle ACB \sim \triangle BDA$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Each $\triangle$ is a rt. $\triangle$. | 1. Why? |
| 2. In $\triangle$s $ACB$ and $BCD$,<br>$\angle C = \angle C$. | 2. Iden. |
| 3. $\therefore \triangle BCD \sim \triangle ACB$. | 3. (306). |
| 4. In $\triangle$s $ACB$ and $BDA$,<br>$\angle A = \angle A$. | 4. Iden. |
| 5. $\therefore \triangle ACB \sim \triangle BDA$. | 5. (306). |
| 6. $\therefore \triangle BCD \sim \triangle ACB \sim \triangle BDA$. | 6. (308). |

---

Ex. **1.** In the figure for Prop. XXX, prove

$$CB \times BD = BA \times CD.$$

Ex. **2.** In same figure if $CD = 9$ and $DA = 16$, find $BD$.

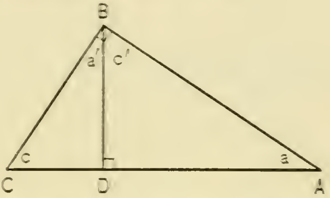
Ex. **3.** Find $BC$ and $BA$ if $CD = 4$ and $DA = 9$. (Use fig. for Prop. XXX.)

Ex. **4.** In the same figure, show that

$$BC \times BA = BD \times CA.$$

## PROPOSITION XXXI. THEOREM

**326.** *In a right triangle, the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse, and either arm is the mean proportional between the hypotenuse and the segment adjacent to that arm.*



**Given** in the rt. △$ABC$, $BD$ the altitude upon the hypotenuse $AC$.

**To prove** (1) $CD : DB = DB : DA$.
(2) $CA : BA = BA : DA$.
(3) $CA : CB = CB : CD$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** $\triangle ABC \sim \triangle BCD \sim \triangle BDA$. | **1.** Prop. 30. |
| **2.** Since $\triangle BCD \sim \triangle BDA$,<br>$\therefore CD : DB = DB : DA$. | **2.** (303). |
| **3.** Since $\triangle ABC \sim \triangle BDA$,<br>$\therefore CA : BA = BA : DA$. | **3.** (303). |
| **4.** Since $\triangle ABC \sim \triangle BCD$,<br>$\therefore CA : CB = CB : CD$. | **4.** (303). |

NOTE. $\angle a$ or $\angle A = \angle a'$ and $\angle c$ or $\angle C = \angle c'$. (Why?)

---

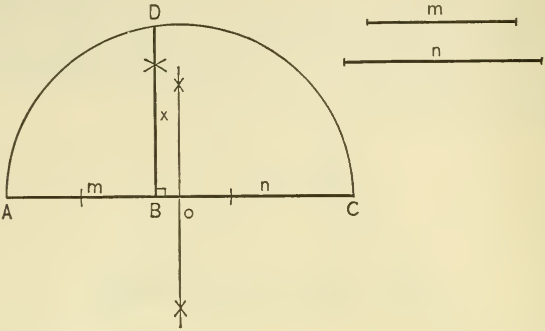
Ex. **1.** In the diagram for Prop. XXXI, if $AD = 4$ and $AC = 9$, what is the length of $AB$?

Ex. **2.** In the same diagram, if $AB = 15$, $AC = 25$, find $AD$ and $BD$.

Ex. **3.** In the same diagram, if $BD = 24$, $AD = 18$, find $AC$ and $BC$.

## PROPOSITION XXXII. PROBLEM

**327.** *To construct the mean proportional between two given lines.*



**Given** lines $m$ and $n$.

**Required** the mean proportional between $m$ and $n$.

**Construction.** Draw $AB = m$, and produce $AB$ to $C$ so that $BC = n$.

On $AC$ as a diameter, describe a semicircle.

At $B$, erect a perpendicular upon $AC$, meeting the circle in $D$.

$BD$ is the required mean proportional.

[The proof is left to the student. Draw $AD$, $DC$. Apply Prop. XXXI.]

**328.** *The perpendicular from any point in the circle upon a diameter is the mean proportional between the segments of the diameter; and the chord joining the point to either extremity is the mean proportional between the diameter and the adjacent segment.*

---

Ex. **1.** If $a$ and $b$ are given lines, construct $\sqrt{ab}$.

Ex. **2.** Construct $\sqrt{6\,ab}$ if $a$ and $b$ are two given lines.

Ex. **3.** Construct a line equal to $a\sqrt{2}$ if $a$ is a given line.

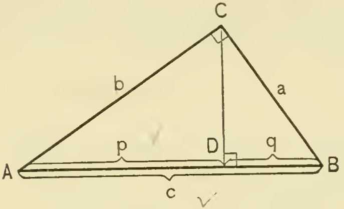
HINT. $a\sqrt{2} = \sqrt{(2\,a)\cdot a}$.

Ex. **4.** Construct a line equal to $a\sqrt{5}$ if $a$ is a given line.

Ex. **5.** Construct a line equal to $a\sqrt{\frac{2}{3}}$ if $a$ is a given line.

## PROPOSITION XXXIII. THEOREM

**329.** *The sum of the squares of the arms of a right triangle is equal to the square of the hypotenuse.*



**Given** $ABC$ a rt. △, having its rt. ∠ at $C$.

**To prove** $a^2 + b^2 = c^2$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $CD \perp AB$, | **1.** Why? |
| **2.** $p : b = b : c$. | **2.** (326). |
| **3.** $\therefore b^2 = c \cdot p$. | **3.** (274). |
| **4.** Similarly $a^2 = c \cdot q$. | **4.** Why? |
| **5.** $a^2 + b^2 = c \cdot p + c \cdot q$<br>$= c(p + q)$. | **5.** Why? |
| **6.** $a^2 + b^2 = c^2$. | **6.** (Sub.) |

**330.** COR. *The square of either arm of a right triangle is equal to the square of the hypotenuse, diminished by the square of the other arm.*

---

EX. **1.** Find the hypotenuse of a right triangle whose arms are respectively

(*a*) 1 ft. and 5 in. (*b*) $m$ and $n$.

EX. **2.** The hypotenuse of a right triangle is 25, one arm equals 20; find the other arm.

EX. **3.** Find the altitude of an equilateral triangle whose side is equal to 8 in.

EX. **4.** *Find the altitude of an equilateral triangle whose side is equal to b.* $(h = \frac{b}{2}\sqrt{3}.)$

Ex. **5**. Find the altitude of an isosceles triangle whose base equals 8 and whose arms equal 5.

Ex. **6**. If the hypotenuse of an isosceles right triangle equals 8 in., what is the length of an arm?

Ex. **7**. The radii of two circles are respectively 6 in. and 21 in., and the distance between their centers is 25 in.; find the length of the common external tangent.

Ex. **8**. Find the side of an equilateral triangle whose altitude equals 10.

Ex. **9**. The squares of the two arms of a right triangle have the same ratio as the segments adjacent to the hypotenuse.

Ex. **10**. If $AD$ is an altitude of a triangle $ABC$,

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Ex. **11**. If the diagonals of a quadrilateral are perpendicular to each other, then the sum of the squares of two opposite sides is equal to the sum of the squares of the other two.

Ex. **12**. *If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.*

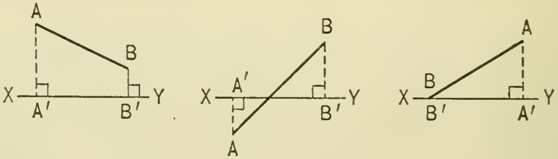
Hint. Draw a rt. △ whose arms are equal respectively to the arms of the given △, and prove the equality of the two △s.

*Ex. **13**. If the square of one side of a triangle is greater than the sum of the squares of the other two, the triangle is obtuse.

Hint. Compare the △ with a rt. △ which has the same arms as the given one.

**331**. Def. The **projection of a point upon a line** is the foot of the perpendicular from the point to the line.

**332**. Def. The **projection of one line upon another** is the segment between the projections of the extremities of the first line upon the second.

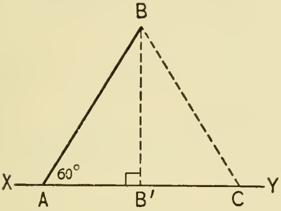


Thus, if $AA' \perp XY$ and $BB' \perp XY$, $A'$ is the projection of $A$ upon $XY$, and $A'B'$ is the projection of $AB$ upon $XY$.

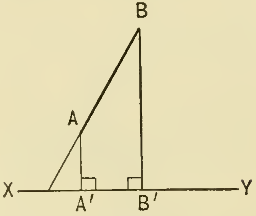
Ex. 1. In acute triangle $ABC$, draw the projection of $AB$ upon $AC$, of $AB$ upon $BC$, of $AC$ upon $AB$.

Ex. 2. If $AB \parallel XY$, prove that the projection of $AB$ upon $XY$ equals $AB$.

Ex. 3. If the side of an equilateral triangle equals 10 in., what is the length of the projection of one side upon another?



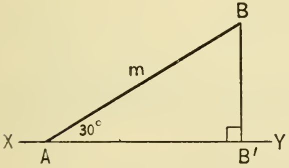
Ex. 4



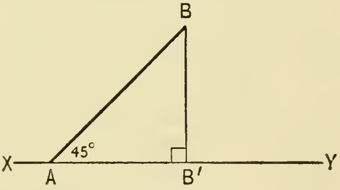
Ex. 5

Ex. 4. If the lines $AB$ and $XY$ include an angle of 60°, the projection of $AB$ upon $XY$ equals one half $AB$ (165).

Ex. 5. Prove that the projection $A'B'$ of $AB$ upon $XY$ equals one half $AB$, if the *prolongation* of $BA$ forms an angle of 60° with $XY$.



Ex. 6



Ex. 7

Ex. 6. If the lines $AB$ and $XY$ include an angle of 30°, and $AB = m$, prove that the projection of $AB$ upon $XY$ equals $\frac{m}{2}\sqrt{3}$.

HINT. $BB' = \frac{m}{2}$. (Ex. 4.)

Ex. 7. If the lines $AB$ and $XY$ include an angle of 45°, and $AB = m$, prove that the projection of $AB$ upon $XY = \frac{m}{2}\sqrt{2}$.

HINT. If $AB' = x$, then $BB' = x$. $\therefore x^2 + x^2 = m^2$.

Ex. **8.** Prove the last two exercises, if the *prolongation* of $AB$ forms an angle of 30°; an angle of 45°. (Diagram similar to Ex. 5.)

Ex. **9.** Find the projection of $AB$ upon $XY$ if $AB = m$, and the angle included by $AB$ and $XY$ equals 120°.

Ex. **10.** Find the projection of $AB$ upon $XY$ if $AB = m$, and the angle included by $AB$ and $XY$ equals 135°.

Ex. **11.** Find the projection of $AB$ upon $XY$ if $AB = m$, and the angle included by $AB$ and $XY$ equals 150°.

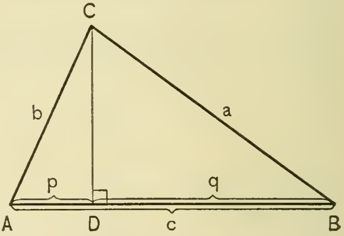
Ex. **12.** If in triangle $ABC$, $AB = 8$, $AC = 10$, and $\angle A = 60°$, find the projection of $AB$ upon $AC$, of $BC$ upon $AC$.

Ex. **13.** If in triangle $ABC$, $AB = 10$ $AC = 12$, and $\angle A = 45°$, find the projection of $AB$ upon $AC$.

Ex. **14.** In the same figure find the projection of $BC$ upon $AC$.

Ex. **15.** In triangle $ABC$, $AC = 24$, $BC = 10$, and $\angle C = 90°$. Find the projection of $AC$ upon $AB$.

**333.** NOTE. $\triangle abc$ denotes a triangle whose sides are $a$, $b$, and $c$. $p$ denotes the projection of $b$ upon $c$, and $q$ the projection of $a$ upon $c$. The other notations used in the following propositions are in accordance with § 242.



Ex. **16.** In $\triangle abc$, if $b = 4$, $p = 2$, find $\angle A$ and $h_c$.

Ex. **17.** In $\triangle abc$, if $b = 5$, $h_c = 4$, $c = 8$, find $q$.

Ex. **18.** In $\triangle abc$, if $b = 10$, $h_c = 8$, and $a = 17$, find $c$.

Ex. **19.** In $\triangle abc$, if $b = 10$, $h_c = 8$, $c = 14$, find $a$.

Ex. **20.** In $\triangle abc$, express $a$ in terms of $b$, $h_c$, and $c$.

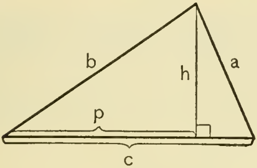
Ex. **21.** In $\triangle abc$, if $a = 20$, $b = 37$, $q = 16$, find $p$.

Ex. **22.** In $\triangle abc$, if $b = 15$, $p = 9$, and $c = 25$, find $a$.

Ex. **23.** In $\triangle abc$, express $a$ in terms of $b$. $c$, and $p$.

## PROPOSITION XXXIV. THEOREM

**334.** *In any triangle, the square of a side opposite an* **acute** *angle is equal to the sum of the squares of the other two sides* **diminished** *by twice the product of one of those sides and the projection of the other side upon it.*



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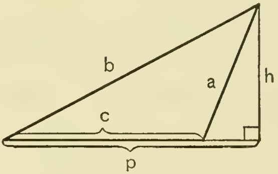


II

**Given** in $\triangle abc$, $p$ the projection of $b$ upon $c$, and the angle opposite $a$ an acute angle.

**To prove** $\quad a^2 = b^2 + c^2 - 2\,cp.$

**Proof.** Denote the perpendicular upon $c$ by $h$.

In diagram I,

$$a^2 = h^2 + (c - p)^2,$$

but $\quad h^2 = b^2 - p^2.$

[Substitute and simplify.]

In diagram II,

$$a^2 = h^2 + (p - c)^2,$$

but $\quad h^2 = b^2 - p^2.$

[To be completed by the student.]

**335.** REMARK. The equation $a^2 = b^2 + c^2 - 2\,cp$ contains four quantities. Therefore **any one of them may be found by algebraical methods if the other three are given.** (Similarly in the following propositions.)

---

Ex. **1.** In $\triangle abc$, find $a$ if

(*a*) $b = 8$, $c = 5$, $p = 4$.

(*b*) $b = 24$, $c = 9$, $p = 12$.

(*c*) $b = 5$, $c = 6$, $p = 3$.

(*d*) $b = 13$, $c = 14$, $p = 12$.

(*e*) $b = 17$, $c = 9$, $p = 15$.

Ex. **2**. If two sides of a triangle equal 15 and 25 respectively, and the projection of 15 upon 25 equals 9, what is the value of the third side?

Ex. **3**. In $\triangle abc$, find $a$ if

(a) $b = 10$, $c = 16$, $\angle A = 60°$. (d) $b = 48$, $c = 13$, $\angle A = 60°$.
(b) $b = 14$, $c = 30$, $\angle A = 60°$. (e) $b = 4$, $c = 3$, $\angle A = 30°$.
(c) $b = 9$, $c = 24$, $\angle A = 60°$. (f) $b = 2$, $c = 3$, $\angle A = 45°$.

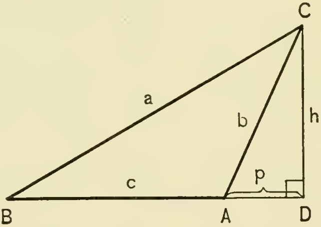
Ex. **4**. The sides of a triangle are 13, 14, and 15. Find the projection of 13 upon 14.

Ex. **5**. The sides of a triangle are 5, 7, and 8. Find the projection of 8 upon 5.

Ex. **6**. The sides of a triangle are 10, 17, 21. Find the projection of 10 upon 21.

### PROPOSITION XXXV. THEOREM

**336.** *In any obtuse triangle, the square of the side opposite the* **obtuse** *angle is equal to the sum of the squares of the other two sides,* **increased** *by twice the product of one of these sides and the projection of the other side upon it.*



**Given** in $\triangle abc$, $p$ the projection of $b$ upon $c$, and the angle opposite $a$ obtuse.

**To prove** $\boldsymbol{a^2 = b^2 + c^2 + 2\,cp}$.

**Proof.** $a^2 = h^2 + (c + p)^2$.

But $h^2 = b^2 - p^2$.

$\therefore\ a^2 = ?$

[To be completed by the student.]

Ex. **1**. In $\triangle abc$, $b = 6$, $c = 10$, $p = 3$, and $\angle A$ is obtuse; find $a$.

Ex. **2**. In $\triangle abc$, $b = 10$, $c = 9$, $p = 6$, and $\angle A$ is obtuse; find $a$.

Ex. **3**. In $\triangle abc$, find $a$ if

(*a*) $b = 3, c = 5, \angle A = 120°$. (*c*) $b = 16, c = 5, \angle A = 120°$.
(*b*) $b = 8, c = 7, \angle A = 120°$. (*d*) $b = 24, c = 11, \angle A = 120°$.

**337**. Remark. If we consider the projection of one side of a triangle upon another as positive when the projection lies on that line, but as negative when it lies on the prolongation, Props. XXXV and XXXIII become special cases of Prop. XXXIV, and we always have:

$$a^2 = b^2 + c^2 - 2\,cp.$$

To compute the projections of sides of a triangle whose angles are not known, always apply this equation. If the result is negative, the triangle is obtuse.

Ex. **1**. In $\triangle abc$, $a = 20$, $b = 15$, and $c = 7$. Find the projection of $b$ upon $c$. Is the triangle obtuse or acute?

Ex. **2**. In $\triangle abc$, $a = 20$, $b = 15$, $c = 25$. Find the projection of $b$ upon $c$. Is angle $A$ obtuse or acute?

Ex. **3**. The sides of a triangle are 4, 13, and 15. Find the projection of 13 upon 4.

Ex. **4**. If the value of $p$ obtained from the above formula equals zero, what does this result signify?

Ex. **5**. In $\triangle abc$, $a = 15$, $b = 13$, $c = 14$. Find $h_c$.

Ex. **6**. In $\triangle abc$, $a = 17$, $b = 10$, $c = 9$. Find $h_c$.

**338**. Cor. 1. In $\triangle abc$, if $p$ denotes the projection of $b$ upon $c$,

$$p = \frac{b^2 + c^2 - a^2}{2\,c}.$$

**339**. Cor. 2. If $h_c$ denotes the altitude upon $c$,

$$h_c = \sqrt{b^2 - p^2} = \sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2}.$$

This expression can be simplified by algebraical operations:

$$h_c^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2 = \left(b + \frac{b^2 + c^2 - a^2}{2\,c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2\,c}\right)$$

$$= \frac{(2\,bc + b^2 + c^2 - a^2)(2\,bc - b^2 - c^2 + a^2)}{4\,c^2}$$

$$= \frac{[(b + c)^2 - a^2][a^2 - (b - c)^2]}{4\,c^2}$$

$$= \frac{(a + b + c)(b + c - a)(a - b + c)(a + b - c)}{4\,c^2}.$$

Let $a + b + c = 2\,s$, *i.e.* let $s$ denote half the perimeter.

$$b + c - a = 2(s - a).$$

$$a - b + c = 2(s - b).$$

$$a + b - c = 2(s - c).$$

$$h_c^2 = \frac{4\,s(s - a)(s - b)(s - c)}{c^2}$$

or

$$h_c = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

---

Ex. **1**. In $\triangle abc$, find $h_c$, if

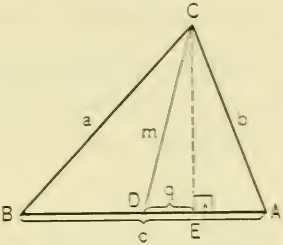
(*a*) $a = 10$, $b = 17$, $c = 21$.
(*b*) $a = 20$, $b = 13$, $c = 21$.
(*c*) $a = 20$, $b = 13$, $c = 25$.
(*d*) $a = 37$, $b = 13$, $c = 40$.

Ex. **2**. The three sides of a triangle are 4, 13, and 15. Find the altitude upon 4.

Ex. **3**. The three sides of a triangle are 25, 30, and 11. Find the altitude upon 11.

### PROPOSITION XXXVI. THEOREM

**340.** *In any triangle, the square of one side plus four times the square of the corresponding median is equal to twice the sum of the squares of the other sides.*



**Given** in $\triangle ABC$, $m_c$ the median to $c$.

**To prove** $2a^2 + 2b^2 = c^2 + 4m_c^2$.

**Proof.** Draw $CE \perp AB$, and suppose $E$ to fall between $A$ and $D$. Let $DE = q$.

$$a^2 = \left(\frac{c}{2}\right)^2 + m^2 + 2\left(\frac{c}{2}\right)q. \qquad (336)$$

$$b^2 = \left(\frac{c}{2}\right)^2 + m^2 - 2\left(\frac{c}{2}\right)q. \qquad (334)$$

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2m^2. \qquad (\text{Ax. } 2.)$$

or

$$2a^2 + 2b^2 = c^2 + 4m^2. \qquad (\text{Ax. } 7.)$$

---

Ex. **1.** The sides of a triangle are 7, 8, and 9 respectively. Find the length of the median to 8.

Ex. **2.** The sides of a triangle are 7, 4, and 9 respectively. Find the length of the median to 9.

Ex. **3.** The sides of a triangle are 10, 5, and 9 respectively. Find the length of the median to 9.

Ex. **4.** The sides of a triangle are 22, 20, and 18 respectively. Find the length of the median to 18.

Ex. **5.** In $\triangle abc$, if $m_c$ denotes the median drawn to $c$, prove that

$$m_c = \frac{1}{2}\sqrt{2a^2 - 2b^2 - c^2}.$$

Ex. **6**. In $\triangle abc$, $a = 8$, $b = 11$, and $m_c = 8\frac{1}{2}$. Find $c$.

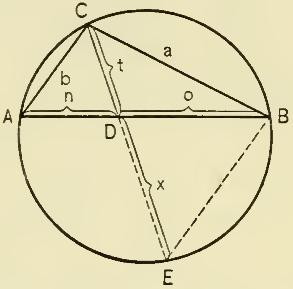
Ex. **7**. In $\triangle abc$, $a = 28$, $c = 32$, $m_c = 38$. Find $b$.

Ex. **8**. Prove Prop. XXXVI if $a = b$.

Ex. **9**. *The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.*

### PROPOSITION XXXVII. THEOREM

**341.** *In any triangle, the product of two sides is equal to the square of the bisector of the included angle plus the product of the segments of the third side.*



**Given** $\triangle abc$, the angle-bisector $t$ dividing $c$ into the segments $n$ and $o$.

**To prove** $ab = t^2 + on.$

**Proof.** Circumscribe a circle about $\triangle abc$.

Produce the angle-bisector $CD$ to meet the circle in $E$. Draw $EB$, and let $DE = x$.

$$\angle ACD = \angle ECB. \qquad \text{(Hyp.)}$$

$$\angle A = \angle E. \qquad (232)$$

$$\triangle ACD \sim \triangle EBC. \qquad (305)$$

$$\therefore\ b : t + x = t : a,$$

or $$ab = t(t + x).$$

$$ab = t^2 + tx.$$

But $$tx = on. \qquad (320)$$

$$\therefore\ ab = t^2 + on.$$

**Ex. 1.** The sides of a triangle are 18, 9, and 21 respectively. Find the length of the bisector corresponding with 21.

HINT. Find $n$ and $o$ by means of (301).

**Ex. 2** The sides of a triangle are 21, 14, and 25 respectively. Find the length of the bisector corresponding with 25.

**Ex. 3.** The sides of a triangle are 22, 11, and 21 respectively. Find the length of the bisector corresponding with 21.

**Ex. 4.** The sides of a triangle are 6, 3, and 7 respectively. Find the length of the bisector corresponding with 7.

**Ex. 5.** In $\triangle abc$, if $t$ denotes the angle bisector of $\angle C$, prove that

$$t^2 = ab - \frac{abc^2}{(a+b)^2}.$$

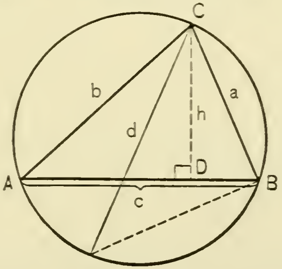
HINT. $$n = \frac{bc}{a+b},\ o = \frac{ac}{a+b}. \qquad (301)$$

* **Ex. 6.** Using the notation and the method of (339), reduce the result of the preceding exercise to the following form

$$t = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

## PROPOSITION XXXVIII. THEOREM

**342.** *In any triangle the product of two sides is equal to the altitude upon the third side, multiplied by the diameter of the circumscribed circle.*



**Given** $d$, a diameter of the circle circumscribed about $\triangle abc$, and $h$, the altitude upon $c$.

**To prove** $ab = hd.$

[The proof is left to the student.]

**343.** Cor. The diameter of the circumscribed circle of any triangle is equal to the product of two sides divided by the altitude upon the third side.

$$\left(d = \frac{ab}{h} \text{ or } d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.\right)$$

---

Ex. **1.** In $\triangle abc$, $a = 6$, $b = 10$, and $h_c = 4$. Find the diameter of the circumscribed circle.

Ex. **2.** In $\triangle abc$, $a = 10$, $b = 15$, $h_c = 6$. Find the radius of the circumscribed circle.

Ex. **3.** $\triangle ABC$ is inscribed in a circle of radius = 5 inches. Find the altitude to $BC$ if $AB = 4$ and $AC = 5$.

Ex. **4.** Find the diameter of the circle circumscribed about $\triangle abc$, if

(*a*) $a = 17$, $b = 8$, $c = 15$.
(*b*) $a = 10$, $b = 17$, $c = 21$.

Ex. **5.** In $\triangle abc$, $a = 20$, $b = 15$, and the projection of $b$ upon $c$ equals 9. Find the radius of the circumscribed circle.

Ex. **6.** In $\triangle abc$, $a = 9$ and $b = 12$. Find $c$ if the diameter of the circumscribed circle equals 15.

## PROBLEMS OF COMPUTATION

Ex. **1.** The arms of a right triangle are 8 and 15 respectively. Compute the hypotenuse and the altitude upon the hypotenuse.

Ex. **2.** A chord 14 in. long is 12 in. distant from the center. Find the radius of the circle.

Ex. **3.** A chord 24 in. long is 5 in. distant from the center. Find the distance of the center from a chord 10 in. long.

Ex. **4.** In $\triangle abc$, $a = 4$, $b = 8$, and the angle opposite to $c$ equals 60°. Find $c$.

Ex. **5.** The shadow of a church steeple upon level ground is 60 ft., while a pole 10 ft. high casts a shadow 3 ft. long. How high is the steeple?

Ex. **6.** Find the product of the segments of a chord drawn through a point 8 in. from the center of a circle whose radius is 10 in. What is the length of the shortest chord that can be drawn through that point?

Ex. **7**. The base of an isosceles triangle is 48 in. Find the altitude if each arm equals 50 in.

Ex. **8**. Two sides and a diagonal of a parallelogram are 7, 9, and 8 respectively. Find the length of the other diagonal.

Ex. **9**. The diagonal of a square is 20 in. Find the side.

Ex. **10**. The sides of a rectangle are 16 and 30 respectively. Find the diagonal.

Ex. **11**. The diameter $AB$ of a circle is produced to $C$, and from $C$ a tangent is drawn to the circle. Find the length of the tangent if $AB = 30$ and $BC = 2$.

Ex. **12**. A ladder 17 ft. long reaches a window 15 ft. high. How far is the lower end of the ladder from the house?

Ex. **13**. In $\triangle abc$, $c = 10$, $\angle A = 30°$, and $\angle C = 90°$. Find $a$ and $b$.

Ex. **14**. The base of an isosceles triangle is $b$, and each arm $a$. Find the altitude.

Ex. **15**. The nonparallel sides $AB$ and $CD$ of a trapezoid are produced till they meet in $E$. Find $AE$ and $BE$ if $AB = 7$ and the bases are 5 and 3 respectively.

Ex. **16**. The diagonals of a rhombus are 10 and 24 in. respectively. Find the perimeter; also, the altitude.

Ex. **17**. Two parallel chords 1 in. apart are respectively 6 in. and 8 in. in length. Find the radius of the circle. (Is there more than one solution?)

Ex. **18**. The altitude of a trapezoid is $h$, the bases $a$ and $b$ respectively. Find the altitudes of the two triangles formed by producing the nonparallel sides until they meet.

Ex. **19**. From a point 24 ft. above sea level the visible horizon has a radius of 6 miles. Find the diameter of the earth.

Ex. **20**. Find the length of the common internal tangent of two circles if the distance between the centers is 17, and the radii are 5 and 3 respectively.

Ex. **21**. A chord 30 in. long subtends an angle of 120°. Find the distance of the chord from the center of the circle.

* Ex. **22**. In triangle $ABC$, $AB = BC = 25$, $AC = 30$, and on $AB$ is laid off $AD = 8$. Find the length of $CD$.

* Ex. **23**. The three sides of a triangle are 14, 16, and 6. Find the angle opposite 14.

* Ex. **24**. In a quadrilateral $ABCD$, $AB = 10$, $BC = 17$, $CD = 13$, $DA = 20$, and $AC = 21$. Find the diagonal $BD$.

## PROBLEMS OF CONSTRUCTION

Ex. **1**. Divide each side of a triangle internally into two parts proportional to the other two sides.

To construct a triangle, having given:

Ex. **2**. $a$, $b$, and $b : c = 4 : 5$.

Ex. **3**. From a given rectangle to cut off a similar rectangle by a line parallel to one of its sides.

Ex. **4**. In a given circle, to inscribe a triangle similar to a given triangle.

Ex. **5**. About a given circle, to circumscribe a triangle similar to a given triangle.

Ex. **6**. Construct a triangle similar to a given triangle and having a given altitude.

Ex. **7**. *To inscribe a square in a given triangle.*

Ex. **8**. Assuming an arbitrary unit, construct a line equal to ($a$) $\sqrt{2}$, ($b$) $\sqrt{3}$, ($c$) $1 + \sqrt{5}$.

Ex. **9**. Construct a line that shall be to a given line as $1 : \sqrt{2}$.

Ex. **10**. Construct a line that shall be to a given line as $\sqrt{3} : 1$.

Ex. **11**. In a given line $AB$, to find a point $C$ such that $AC : BC = 1 : \sqrt{2}$.

Ex. **12**. To construct a parallelogram similar to a given parallelogram and having a given diagonal.

Ex. **13**. *To construct a triangle similar to a given triangle and having a given perimeter.*

Ex. **14**. If $a$ and $b$ are two given lines, construct a line equal to $\frac{2a^2}{b}$.

Ex. **15**. From a point without a circle, to draw a secant whose external segment is equal to one half the secant. (See hint for Ex. 16.)

Ex. **16**. To construct a circle, touching a given circle in a given point, and touching a given line. HINT. Draw a tangent to the circle.

Ex. **17**. To construct a circle, touching two parallel lines and passing through a given point.

Ex. **18**. In a given circle, to inscribe a rectangle, having given the ratio of two sides.

* Ex. **19**. To divide a trapezoid into two similar trapezoids by a line parallel to the bases.

## THEOREMS

Ex. **1**. If a chord is bisected by another, either segment of the first is a mean proportional between the segments of the other.

Ex. **2**. If in the triangle $ABC$ the altitudes $BD$ and $AE$ meet in $F$, and $AB = BC$, then

$$BC : AF = BD : CD.$$

Ex. **3**. If between two parallel tangents, a third tangent is drawn, the radius is the mean proportional between the segments of the third tangent.

Ex. **4**. If two circles are tangent externally, and through the point of contact a secant is drawn, the chords formed are proportional to the radii.

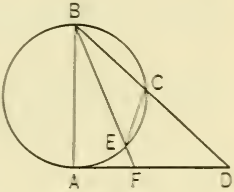
Ex. **5**. If $C$ is the mid-point of the arc $AB$, and a chord $CD$ meets the chord $AB$ in $E$, then

$$CE : CA = CA : CD.$$

Ex. **6**. If $AB$ is a diameter, $AD$ a tangent, and $FB$ and $DB$ are secants, prove that

$$BE \times BF = BC \times BD.$$

Ex. **7**. Prove that in any right triangle the sum of the hypotenuse and the diameter of the inscribed circle is equal to the sum of the arms of the triangle.



Ex. 6

Ex. **8**. If two circles intersect, their common chord produced bisects the common tangents.

Ex. **9**. If an isosceles triangle is inscribed in a circle, the tangents drawn at the vertices form another isosceles triangle.

Ex. **10**. The tangents drawn at the vertices of an inscribed rectangle inclose a rhombus.

Ex. **11**. *Two parallelograms are similar when they have an angle of the one equal to an angle of the other, and the including sides proportional.*

Ex. **12**. *Two rectangles are similar if two adjacent sides are proportional.*

Ex. **13**. Two circles are tangent externally, and through the point of contact two straight lines are drawn terminating in the circles; prove that the corresponding segments of the lines are proportional.

## MISCELLANEOUS

Ex. **1**. *If two parallel lines are cut by three or more transversals passing through a common point, the corresponding segments are proportional.*

Ex. **2**. *Prove the converse of Ex. 1.*

Ex. **3**. *The median to a side of a triangle bisects every line parallel to this side and limited by the other sides.*

Ex. **4**. Construct an isosceles triangle given the base and the radius of the inscribed circle.

Ex. **5**. Prove that the right angle of any right triangle is bisected by a line drawn from its vertex to the point of intersection of the diagonals of the square constructed on the hypotenuse.

Ex. **6**. The bisectors of the equal angles of an isosceles triangle meet the equal sides at points $D$ and $E$; prove that $DE$ is parallel to the base of the triangle.

Ex. **7**. Construct a right triangle, given the segments made on the hypotenuse by the bisector of the right angle.

Ex. **8**. The perpendiculars from the mid-points of two sides of a triangle to the third side are equal.

Ex. **9**. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular to each other.

Ex. **10**. Show how to construct a tangent to a circle at a given point on the circle, without using the center.

Ex. **11**. The radius of a circle is 30 feet. Find the length of a tangent drawn to the circle from a point 4 feet from the circle.

Ex. **12**. A rectangle is inscribed in a circle of radius 5 feet. The base is twice the altitude. Find the perimeter of the rectangle.

Ex. **13**. Two circles of radii 12 in. and 9 in. respectively have their centers 28 in. apart. How far from the center of each circle is the line of centers cut by a common external tangent?

Ex. **14**. The bases of a trapezoid are $8''$ and $12''$ respectively. The altitude is $3''$. How long is the line drawn between the legs and parallel to the bases, if it is $2''$ from the lower base?

# BOOK IV

## AREAS OF POLYGONS

**344.** Def. The **unit of surface** is a square whose side is the unit of length, etc.

Thus, a square 1 in. long and 1 in. wide is a square inch. Or a square 1 yd. long and 1 yd. wide is a square yard.

**345.** The **area of a surface** is the number of units of surface it contains.

Thus, if the floor of a room is 25 ft. long and 15 ft. wide, it contains 15 × 25 or 375 sq. ft. Hence the area of the floor is 375 sq. ft.

**346.** Two figures are **equivalent** or **equal** if their areas are equal.

Thus, if the area of $\triangle ABC = 25$ sq. ft., and the area of $\square MNOP = 25$ sq. ft., then $\triangle ABC$ is *equivalent to* $\square MNOP$, or in symbols:

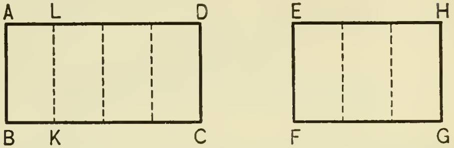
$$\triangle ABC = \square MNOP.$$

Note. The symbol $=$ refers to the size of figures, while the symbol $\sim$ relates to their shape. The symbol of congruence $\cong$ relates to both size and shape. The symbol $\sim$ has no meaning for figures that cannot differ in shape, *e.g.* for straight lines. Thence the symbol of congruence, $\cong$ when applied to straight lines has the same significance as the symbol of equality $=$.

If the symbol of equality $=$ refers to areas, it may read either "is equivalent to" or "equals." Since many authors, however, designate congruent figures as *equal figures,* confusion may be avoided by giving preference to the term equivalent. The use of a particular symbol for equivalence $\backsimeq$ cannot be recommended.

### PROPOSITION I. THEOREM

**347.** *Rectangles having equal altitudes are to each other as their bases.*



**Given** rectangles $ABCD$ and $EFGH$ with bases $BC$ and $FG$.

**To prove** $ABCD : EFGH = BC : FG.$

CASE I (Commensurable)

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Take $BK$ as unit of measure for the bases. Then $BC = m(BK)$, $FG = n(BK)$. | **1.** The bases are commensurable. |
| **2.** $\frac{BC}{FG} = \frac{m}{n}.$ | **2.** (224). |
| **3.** At the points of division on $BC$ and $FG$, erect $\perp$s. The rectangles will be cut up into $m$ and $n$ small rectangles, each equal to $ABKL$. | **3.** (47, 141, 147). |
| **4.** $\frac{ABCD}{EFGH} = \frac{m}{n}.$ | **4.** (224). |
| **5.** $\therefore \frac{ABCD}{EFGH} = \frac{BC}{FG}.$ | **5.** Why? |

CASE II. (Incommensurable.) See Appendix.

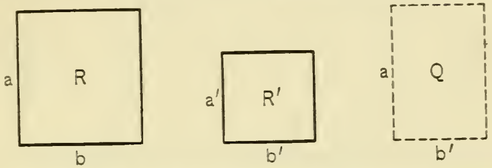
**348.** COR. *Rectangles having equal bases are to each other as their altitudes.*

EX. To divide a given rectangle into three congruent rectangles.

[Practical Applications, page 296.]

## PROPOSITION II. THEOREM

**349.** *The areas of two rectangles are to each other as the products of their bases and altitudes.*



**Given** rectangles $R$ and $R'$, having the bases $b$ and $b'$ and the altitudes $a$ and $a'$ respectively.

**To prove** $$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Construct the rectangle $Q$, having the same base as $R'$, and the same altitude as $R$. | **1.** To construct a rectangle, given base and altitude. |
| **2.** $\frac{R}{Q} = \frac{b}{b'}$. | **2.** Prop. I. |
| **3.** $\frac{Q}{R'} = \frac{a}{a'}$. | **3.** (348). |
| **4.** $\frac{R}{R'} = \frac{a \times b}{a' \times b'}$. | **4.** Why? |

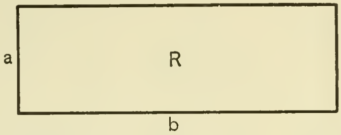
Ex. **1.** Find the ratio of a rectangle 4 by 5 ft., and a square having a side of 10 ft.

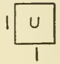
Ex. **2.** The diagonal of a rectangle is 26 in. long, and one of its sides 24 in. The diagonal of another rectangle is 25 in. long, and one of its sides 20 in. Find the ratio of the areas of the two rectangles.

Ex. **3.** The ratio of the areas of two rectangles is as 8 is to 3. The dimensions of the first are 6 and 12. Find the altitude of the second rectangle, if its base is 9.

### PROPOSITION III. THEOREM

**350.** *The area of a rectangle is equal to the product of its base and altitude.*





**Given** $R$ a rectangle with base $b$ and altitude $a$.

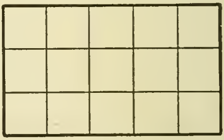
**To prove** Area of $R = a \times b$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Let $U$ be the unit of surface. | 1. Why? |
| 2. $\frac{R}{U} = \frac{a \times b}{1 \times 1}$. | 2. (349). |
| 3. But $\frac{R}{U}$ is the area of $R$. | 3. (345). |
| 4. $\therefore$ area of $R = a \times b$. | 4. Why? |

**351.** COR. *The area of a square is equal to the square of a side.*

**352.** REMARK. If the base and altitude of a rectangle are expressed by integral numbers, Prop. III may be proved by dividing the figure into squares.



Thus, if the base contains 5 and the altitude 3 linear units, the figure can be divided into fifteen squares, each being the unit of surface.

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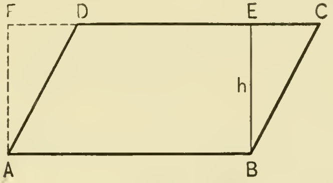
Ex. **1.** A rectangular field is 24 yd. long and 15 yd. wide. Find the area.

Ex. **2.** The area of a rectangle is 360 sq. in., and its base is 4 yd. Find the altitude.

Ex. **3.** A rectangle has an area of 125 sq. in. and is five times as long as wide. Find the dimensions of the rectangle.

### PROPOSITION IV. THEOREM

**353.** *The area of a parallelogram is equal to the product of its base and its altitude.*



**Given** parallelogram $ABCD$ having the base $AB = b$, and the altitude $BE = h$.

**To prove** $ABCD = b \times h$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $AF \perp$ to $AB$ to meet $CD$ produced at $F$. | 1. Why? |
| 2. $AF \parallel BE$. | 2. (96). |
| 3. $ABEF$ is a rectangle, with base $b$ and alt. $h$. | 3. Why? |
| 4. $ABCD$ is a ▱ with base $b$ and alt. $h$. | 4. Hyp. |
| 5. $AF = BE$, $AD = BC$. | 5. Why? |
| 6. Rt. $\triangle AFD \cong$ rt. $\triangle BEC$. | 6. Hy. a. = hy. a. |
| 7. $ABCF = ABCF$. | 7. Iden. |
| 8. ▱$ABCD$ = Rect. $ABEF$. | 8. Equals from equals ⋯. |
| 9. But Rect. $ABEF = bh$. | 9. Why? |
| 10. $\therefore$ ▱$ABCD = bh$. | 10. Sub. |

**354.** COR. 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

**355.** COR. 2. *Any two parallelograms are to each other as the products of their bases and altitudes.*

**356.** Cor. 3. *Parallelograms having equal bases are to each other as their altitudes.*

**357.** Cor. 4. *Parallelograms having equal altitudes are to each other as their bases.*

---

Ex. **1**. Find the area of a parallelogram whose base is 15 in. and whose altitude is 2 ft.

Ex. **2**. Two sides of a parallelogram are 15 and 20 respectively, and include an angle of 30°. Find the area.

Ex. **3**. The sides of a parallelogram are 5 and 8 respectively, and the projection of 5 upon 8 is 3. Find the area.

Ex. **4**. To divide a parallelogram into three equivalent parts.

Ex. **5**. The sides of a parallelogram are 13 and 14, and one diagonal equals 15. Find the area.

Ex. **6**. The sides of a parallelogram are 4 in. and 5 in. and the angle included by them equals 45°. Find the area.

Ex. **7**. The area of a parallelogram is 144 sq. inches. Find its altitude in inches, if the base is 3 feet long.

Ex. **8**. A parallelogram has its base equal to 30 feet and its altitude equal to 12 feet. Find the altitude of an equivalent rectangle having a base of 20 feet.

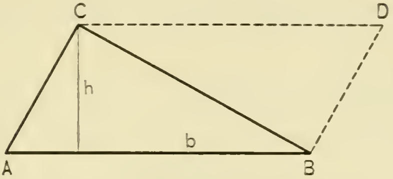
Ex. **9**. A rectangle is 12 by 20. Find the base and a side of an equivalent oblique parallelogram having an angle of 30°, if the base of the parallelogram is 4 units longer than a side.

Ex. **10**. The bases of two parallelograms are to each other as 3 is to 2. The area of the first is to that of the second as 5 is to 3. Compare their altitudes.

Ex. **11**. What change occurs in the area of a parallelogram: (*a*) when the base remains constant but the altitude increases (or decreases); (*b*) when the angle between the base and a side changes but the base and side remain constant?

## PROPOSITION V. THEOREM

**358.** *The area of a triangle is equal to one half the product of its base and altitude.*



**Given** $\triangle ABC$, having base $b$ and altitude $h$.

**To prove** $\triangle ABC = \frac{1}{2} b \times h$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $BD \parallel AC$, and $CD \parallel AB$. | **1.** Why? |
| **2.** $ABDC$ is a ▱. | **2.** Why? |
| **3.** ▱$ABDC = bh$. | **3.** (353). |
| **4.** $\triangle ABC = \frac{1}{2}$ ▱$ABDC$. | **4.** (140). |
| **5.** $\frac{1}{2}$ ▱$ABDC = \frac{1}{2} bh$ | **5.** Why? |
| **6.** $\therefore \triangle ABC = \frac{1}{2} bh$. | **6.** Why? |

**359.** COR. 1. *Triangles having equal bases and equal altitudes are equivalent.*

**360.** COR. 2. *Any two triangles are to each other as the products of their bases and altitudes.*

**361.** COR. 3. *Triangles having equal bases are to each other as their altitudes.*

**362.** COR. 4. *Triangles having equal altitudes are to each other as their bases.*

Ex. **1**. Prove: *If two triangles have a common base and their vertices lie in a line parallel to the base, the triangles are equivalent.*

Ex. **2**. If $A$ denotes the area, and $2s$ the perimeter of $\triangle abc$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)}. \qquad (339)$$

Ex. **3**. The area of a triangle is 600 sq. in., and the altitude is 20 in. Find the base.

Ex. **4**. Two sides of a triangle are 5 and 8 respectively, and include an angle of 30°. Find the area.

Ex. **5**. Find the area of a triangle whose sides are respectively

(*a*) 13, 14, 15.
(*b*) 9, 10, 17.
(*c*) 11, 25, 30.
(*d*) 4, 13, 15.

Ex. **6**. Find the area of $\triangle abc$, if $a = 10$, $b = 17$, $h_c = 8$. (2 sol.)

Ex. **7**. *The diagonals of a parallelogram divide the parallelogram into four equivalent triangles.*

Ex. **8**. Two triangles are equivalent if two sides of the one are equal respectively to two sides of the other, and the included angles are supplementary.

Ex. **9**. What is the locus of the vertices of all equivalent triangles constructed on a common base?

Ex. **10**. Find the altitude of a triangle, with a base of 8 in., that is equivalent to a parallelogram with a base of 14 in. and an altitude of 6 in.

Ex. **11**. Find the area of an equilateral triangle having each side equal to 4 in.

Ex. **12**. Find the area of an isosceles triangle whose base is 20 feet and whose vertex angle is 120°.

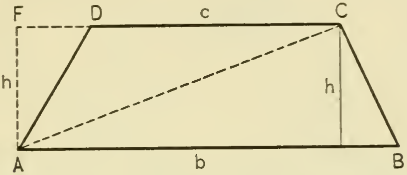
Ex. **13**. Find the area of an isosceles right triangle having its hypotenuse equal to 18 yards.

Ex. **14**. Find the area of an equilateral triangle if its altitude is $4\sqrt{3}$.

Ex. **15**. What is the area of an isosceles triangle if an arm is 20 feet long and its base is 32 feet long?

## PROPOSITION VI. THEOREM

**363.** *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*



**Given.** Trapezoid $ABCD$ has the bases $b$ and $c$ respectively, and the altitude $h$.

**To prove** $ABCD = \frac{1}{2}h(b + c)$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $AC$; also, $AF \perp CD$ produced. | **1.** Why? |
| **2.** $AF = h$. | **2.** Why? |
| **3.** $\triangle ADC = \frac{1}{2}c \times AF$. | **3.** (358). |
| **4.** $\triangle ADC = \frac{1}{2}hc$. | **4.** Sub. |
| **5.** $\triangle ACB = \frac{1}{2}hb$. | **5.** (358). |
| **6.** $\therefore ABCD = \frac{1}{2}hb + \frac{1}{2}hc$,<br>or $= \frac{1}{2}h(b + c)$. | **6.** Why? |

Ex. **1.** *The area of a trapezoid equals the product of its altitude and its median.*

Ex. **2.** The line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equivalent trapezoids.

Ex. **3.** The bases of a trapezoid are 12 and 8 respectively, and the altitude is 5. Find the area.

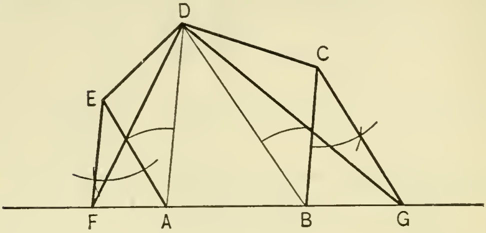
Ex. **4.** In the diagram for Prop. VI, if $b = 6$, $c = 4$, $BC = 10$, and $\angle B = 30°$. Find the area.

Ex. **5.** The area of a trapezoid is 200, the bases are 15 and 25 respectively. Find the altitude.

Ex. **6.** The area of a trapezoid is 30, the altitude is 5, and one base is 8. Find the other base.

### PROPOSITION VII. PROBLEM

**364.** *To transform a polygon into a triangle.*[1]



**Given** polygon $ABCDE$.

**Required** to construct a △ equivalent to $ABCDE$.

**Construction.** Draw $DB$ and $DA$. Through $C$ and $E$ draw $CG$ and $EF$ parallel respectively to $DB$ and $DA$. Side $AB$ produced will intersect these parallels at $F$ and $G$. The required △ is $FDG$.

**Proof.** $\triangle DBG = \triangle BCD$ and $\triangle DFA = \triangle DEA$. (Why?). Polygon $ABCDE = \triangle BCD + \triangle BDA + \triangle DEA = \triangle FDG$. (Why?)

Ex. **1.** Transform a square into a right triangle.

Ex. **2.** Transform a rectangle into a triangle.

Ex. **3.** Transform a quadrilateral into a triangle; also a trapezoid into a triangle.

Ex. **4.** Transform a parallelogram into a rectangle; into a right triangle.

Ex. **5.** Transform an isosceles triangle into a rectangle; also an equilateral triangle into a rectangle.

Ex. **6.** Transform a scalene triangle into an isosceles triangle.

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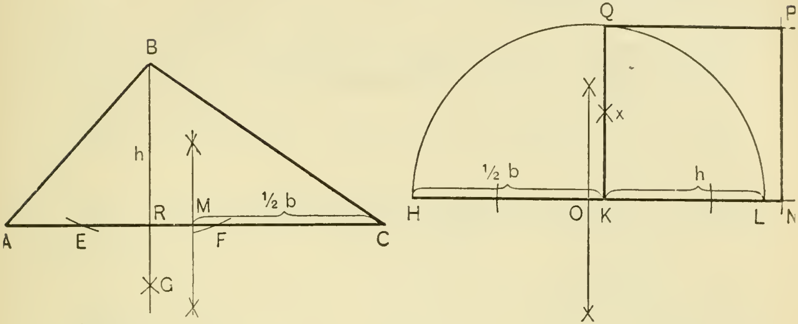
[1] To transform one figure into another means to construct an equivalent figure.

### PROPOSITION VIII. PROBLEM

**365.** *To transform a triangle into a square.*



**Given** $\triangle ABC$.

**Required** to construct a square equivalent to $\triangle ABC$.

**Analysis.** Let $AC$ (the base) $= b$, $h =$ the alt. of $\triangle$. Area of $\triangle = \frac{1}{2}bh$. Let $x =$ a side of the required square. Area of square $= x^2$.

$\therefore x^2 = \frac{1}{2}bh$, or $x \cdot x = \frac{1}{2}b \cdot h$.

Then $\frac{1}{2}b : x = x : h$. Consequently, a side of the square is a mean proportional between the altitude and one half the base of the $\triangle$.

**Construction.** 1. Bisect base of triangle $AC$ to get $MC = \frac{1}{2}b$.

2. Drop $\perp$ from $B$ to base $AC$ and call it $h$.

3. With $\frac{1}{2}b + h$(or $HK + KL$) as a diameter of a circle construct a semicircle.

4. At the point (in diameter) common to $\frac{1}{2}b$ and $h$ (i.e. $K$) construct $\perp$ to $HL$ meeting semicircle in point $Q$. Then on $QK$ (i.e. $x$) construct the square having $x$, or $QK$, as a side. [The proof lies in the analysis given above.]

Ex. Transform in equivalent squares:

(*a*) an equilateral $\triangle$

(*b*) an obtuse parallelogram

(*c*) a trapezoid

(*d*) a quadrilateral

**366.** Method XX. **One of the fundamental methods for transformation of geometric figures is the method that applies the algebraic equation.**

**367.** Note. In the following propositions, $a$, $b$, $c$, $d$, etc., denote given lines, while $x$, $y$, $z$, etc., denote required lines.

**368.** (1) Construct $x = a + b$.

(2) Construct $x = a - b$.

(3) If $m$ denotes a given rational number.

Construct $x = m \cdot a$.

(4) Construct $x = \frac{a}{m}$.

(5) Construct $x = \frac{ab}{c}$.

Hint. $x$ is the fourth proportional to $c$, $a$, and $b$.

(6) Construct $x = \frac{a^2}{b}$. [Use (5).]

(7) Construct $x = \sqrt{ab}$.

Hint. $x$ is the mean proportional between $a$ and $b$ (327).

(8) Construct $x = \sqrt{a^2 + b^2}$.

Hint. $x$ = hy. of rt. △ with arms $a$ and $b$.

(9) Construct $x = \sqrt{a^2 - b^2}$.

Hint. $x$ = one arm of rt. △ with hy. = $a$, arm = $b$.

(10) Construct $x = a\sqrt{2}$.

Hint. $x$ = diag. of a square with side $a$.

(11) If $m$ denotes a known number.

Construct $x = a\sqrt{m}$.

Hint. $x = \sqrt{a(am)}$.

Ex. **1**. Construct $x$ for each equality in section 368. [Let $m = 3$ in (3), (4), and (11).]

Ex. **2**. Construct a square three times as large as a given square.

Ex. **3**. Transform a parallelogram into a rectangle having its base twice that of the parallelogram.

Ex. **4**. Construct a square equivalent to one half a given triangle.

Ex. **5**. Construct $x = a\sqrt{5}$; $x = \frac{1}{2}a\sqrt{3}$; $x = \sqrt{a^2 + ab}$;

$$x = \sqrt{a^2 - ab}; \quad x = \frac{a^2 - b^2}{c}.$$

Ex. **6**. Given the base and a base angle of a triangle, to construct the triangle equivalent to a given trapezoid.

Ex. **7**. Construct a square equivalent to a given trapezoid.

Ex. **8**. Given a rectangle, to construct an equivalent rectangle having its base equal to one half the perimeter of the given rectangle.

* Ex. **9**. Construct an equilateral triangle equivalent to a given square.

**369**. The product of (the numerical measures of) two lines can be represented geometrically by the area of the rectangle formed by these lines. Hence, if the letters $a$, $b$, $c$, etc., represent a number of lines, then any homogeneous expression of the second degree that involves these letters can be represented by the sum (or difference) of rectangles.

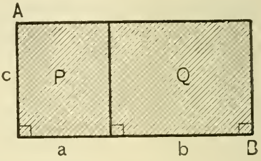
Note. All diagrams of the following exercises may be constructed by drawing first two lines that are perpendicular to each other, and then constructing parallels to these according to the conditions of the theorem.

We may therefore assume in each exercise the following fact, which can easily be proved:

All quadrilaterals in the diagram are rectangles, and their opposite sides are equal.

Ex. If $a$, $b$, $c$, and $d$ are four lines, $a > b$, and $b > c$ represent geometrically: (a) $ab$; (b) $(a + b)b$; (c) $(a - c)d$; (d) $(a + b)(c + d)$; (e) $(a - b)(c + d)$; (f) $ab + ac$; (g) $a^2 + ab + b^2$; (h) $a^2 - b^2$; (i) $(a - b)^2$.

**370.** Any homogeneous identity of the second degree can be demonstrated geometrically by showing that the areas which represent the two members of the identity are equal.



Thus to illustrate that

$(a+b)c = ac + bc$ we have,

$(a+b)c =$ area of $AB$.

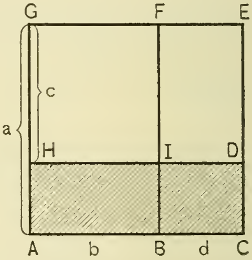
$ac =$ area of $P$.

$bc =$ area of $Q$.

But obviously

$$AB = P + Q.$$

$$(a+b)c = ac + bc.$$



Similarly to show that

$$(a-c)(b+d) = ab + ad - bc - bd$$

$AH = a - c,\ AC = b + d.$

$\therefore$ the left member of the identity, or

$(a-c)(b+d) =$ area of $AD$.[1]

But $a \cdot b = GB,\ a \cdot d = FC.$

$\therefore ab + ad = GC,$

and $bc = HF,\ dc = FD.$

$\therefore ab + ad - bc - dc = AD,$

or $(a-c)(b+d) = ab + ad - bc - dc.$

Ex. Prove geometrically the following algebraic formulæ:[1]

(*a*) $a(b+c+d) = ab + ac + bd.$

(*b*) $(a+b)(c+d) = ac + bc + ad + bd.$

(*c*) $a(b-c) = ab - ac.$

(*d*) $(a+b)^2 = a^2 + 2ab + b^2.$

(*e*) $a^2 - b^2 = (a+b)(a-b).$

(*f*) $(a-b)^2 = a^2 - 2ab + b^2.$ Hint. $a^2 - ab + b^2 - ab.$

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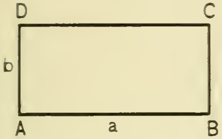
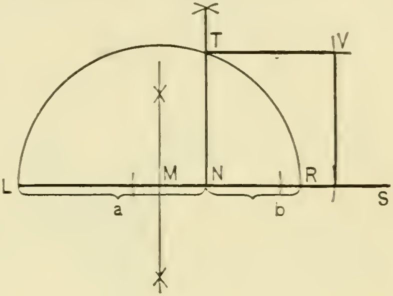
[1] The student is advised in the following exercises to shade the area which represents the left member.

### Proposition IX. Problem

**371.** *To transform a rectangle into a square.*





**Given** rectangle $ABCD$.

**Required** a square $= ABCD$.

**Construction.** On indefinite line $LS$, lay off $LN$ and $NR$ equal to $a$ and $b$ respectively. On $(a + b)$ as a diameter construct a semicircle. At $N$ erect $\perp$ to the diameter meeting the semicircle at $T$. $NT$ equals a side of the required square. Construct the square $NV$.

**Proof.**

$$\text{Area rectangle} = ab.$$

$$\text{Area square} = x^2.$$

$$\therefore x^2 = ab, x = \sqrt{ab}.$$

Consequently, $x$ is a mean proportional between $a$ and $b$.

By the algebraic method construct:

Ex. **1.** A square equivalent to a parallelogram.

Ex. **2.** A square equivalent to twice a triangle.

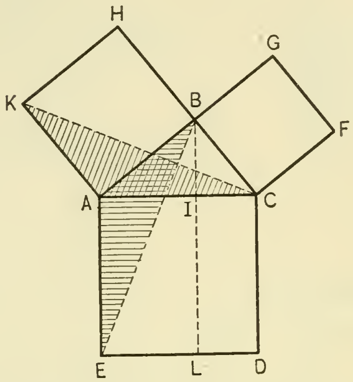
Ex. **3.** A square equivalent to one half a trapezoid.

Ex. **4.** A square equivalent to the sum of two triangles.

Ex. **5.** A square equivalent to the difference between a rectangle and a triangle.

## Proposition X. Theorem

**372.** *In a right triangle the sum of the squares on the arms is equivalent to the square on the hypotenuse.*[1]



**Given** $AH$ and $BF$ squares on the arms, $AD$ the square on the hypotenuse, of the right triangle $ABC$.

**To prove** sq. $AD$ = sq. $AH$ + sq. $BF$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Draw $BL \perp$ to $ED$. | 1. Why? |
| 2. $BL \parallel AE$. | 2. Why? |
| 3. Draw $KC$ and $BE$. | 3. Why? |
| 4. $BA$ = an altitude of $\triangle KAC$.<br>$AI$ = an altitude of $\triangle BAE$. | 4. Why? |
| 5. $KA = AB$, and $AC = AE$. | 5. Hyp. |
| 6. $\angle KAB = \angle EAC$. | 6. Rt. ∡ are =. |
| 7. $\angle BAC = \angle BAC$. | 7. Iden. |
| 8. $\angle KAC = \angle BAE$. | 8. Why? |

---

[1] The shading in this figure is employed merely to point out the two congruent triangles.

| | |
|---|---|
| **9.** $\triangle KAC \cong \triangle BAE$. | **9.** s. a. s. = s. a. s. |
| **10.** $\triangle KAC = \frac{1}{2}(ABHK)$.<br>$\triangle BAE = \frac{1}{2}(AELI)$. | **10.** Why? |
| **11.** $\therefore$ Square $AH$ = Rect. $AL$. | **11.** Why? |
| **12.** Square $BF$ = Rect. $DI$. | **12.** Steps similar to (1 – 11). |
| **13.** $\therefore$ Sq. $AH$ + Sq. $BF$ = Sq. $AD$. | **13.** Why? |

**373.** Cor. *The square on an arm of a right triangle is equivalent to the difference between the squares on the hypotenuse and the other arm.*

**374.** Note. The preceding theorem was first demonstrated by Pythagoras about 550 B.C., and is called after him the **Pythagorean Theorem.** The proof above was given by Euclid (about 300 B.C.).

## Proposition XI. Problem

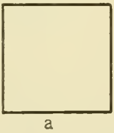
**375.** *To construct a square equivalent to the sum of two given squares.*

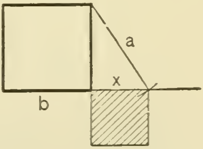
[The solution is left to the student. Apply Prop. X.]

## Proposition XII. Problem

**376.** *To construct a square equivalent to the difference of two given squares.*

[The solution is left to the student.]





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Ex. 1. To construct a square equivalent to the sum of three given squares.

Ex. 2. Construct $x^2$, given $x^2 = a^2 + b^2 - c^2$.

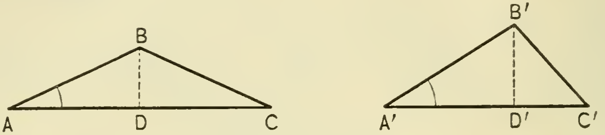
Ex. 3. Construct $x^2$, given $x^2 = 2\,a^2$.

Hint. $x^2 = a^2 + a^2$.

Ex. 4. Using above method, construct a square equivalent to three times a given square.

### PROPOSITION XIII. THEOREM

**377.** *The areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



**Given** $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$.

**To prove** $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \times AC}{A'B' \times A'C'}$.

| | | |
|---|---|---|
| **Proof.** 1. | Draw the altitudes $BD$ and $B'D'$. | 1. Why? |
| 2. | $\angle A = \angle A'$. | 2. Hyp. |
| 3. | $\therefore \triangle ABD \sim \triangle A'B'D'$. | 3. (306). |
| 4. | $\dfrac{BD}{B'D'} = \dfrac{AB}{A'B'}$ | 4. Why? |
| 5. | $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AC \times BD}{A'C' \times B'D'}$ | 5. (360). |
| | $= \dfrac{AC}{A'C'} \times \dfrac{BD}{B'D'}$ | |
| 6. | $= \dfrac{AC}{A'C'} \times \dfrac{AB}{A'B'}$ | 6. Sub. |
| 7. | $= \dfrac{AC \times AB}{A'C' \times A'B'}$. | 7. Why? |

---

Ex. **1.** In triangles $ABC$ and $A'B'C'$, $\angle A = \angle A'$, $AB = 6$ in., $AC = 9$ in., $A'B' = 1$ ft., and $A'C' = 2$ ft. Find the ratio of $\triangle ABC$ to $\triangle A'B'C'$.

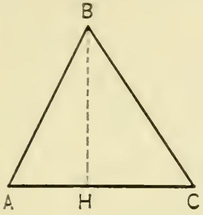
Ex. **2.** Triangle $ABC$ is equivalent to triangle $A'B'C'$, and $\angle A = \angle A'$. Find $AC$ if $AB = 2$ in., $A'B' = 3$ in., and $A'C' = 4$ in.

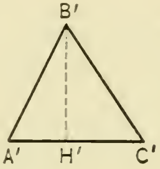
Ex. **3.** If $\triangle ABC = \triangle A'B'C'$, and $\angle A = \angle A'$, then

$$AB : A'B' = A'C' : AC.$$

### PROPOSITION XIV. THEOREM

**378.** *Similar triangles are to each other as the squares of their corresponding sides.*





**Given** $\triangle ABC \sim \triangle A'B'C'$.

**To prove** $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Draw corresponding altitudes $BH$ and $B'H'$. | 1. Why? |
| 2. $\angle A = \angle A'$. | 2. Hyp. |
| 3. Rt. $\triangle ABH \sim$ rt. $\triangle A'B'H'$. | 3. (306). |
| 4. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AC \times BH}{A'C' \times B'H'} = \dfrac{AC}{A'C'} \times \dfrac{BH}{B'H'}$. | 4. (360). |
| 5. $\dfrac{BH}{B'H'} = \dfrac{AC}{A'C'}$. | 5. (314). |
| 6. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AC}{A'C'} \times \dfrac{AC}{A'C'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$. | 6. Sub. |
| 7. In like manner $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$. | 7. Steps (1–6). |

**379**. *Similar triangles are to each other as the squares of their corresponding altitudes.*

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Ex. **1**. Similar triangles are to each other as the squares of their perimeters.

Ex. **2**. The sides of a triangle are 4, 7, and 8. What are the sides of a similar triangle whose area is four times as large?

Ex. **3**. Find the ratio of two similar triangles, two of whose corresponding sides are 2 in. and 3 in. respectively.

Ex. **4**. Compare the areas of two similar triangles if two corresponding sides have the ratio of 2 : 3.

Ex. **5**. The areas of two similar triangles are to each other as 9 : 25. If the perimeter of the first is 36, what is the perimeter of the second?

Ex. **6**. The area of one equilateral triangle is 100 sq. in. What is the area of a second equilateral triangle, if its base is twice the base of the first?

Ex. **7**. A line is drawn connecting the mid-points of two sides of a triangle. What part of the given triangle is the triangle cut off by the given line?

Ex. **8**. What is the ratio of the two right triangles $CBD$ and $BDA$ in the figure on page 186, Prop. XXX?

Ex. **9**. If in Ex. **8**, the two arms are 5 and 12, find the numerical ratio of the two right triangles in that exercise.

Ex. **10**. The nonparallel sides of a trapezoid are prolonged to meet at a point. If the bases of the trapezoid are 20 and 12, what is the ratio of the areas of the triangles in the figure?

Ex. **11**. Similar triangles are to each other as the squares of their corresponding ($a$) medians, ($b$) angle-bisectors.

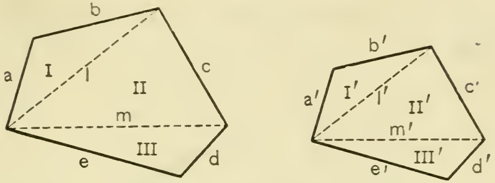
Ex. **12**. Squares are constructed on the base and the altitude of an equilateral triangle. Compare their areas.

Ex. **13**. The areas of two similar triangles are 64 and 225 respectively. The altitude of the first triangle is 6. What is the length of the corresponding altitude of the other?

Ex. **14**. The bases of a trapezoid are 12 and 8 respectively. The nonparallel sides are prolonged until they meet. If the area of the trapezoid is 90, what is the area of the smaller triangle?

### PROPOSITION XV. THEOREM

**380.** *Similar polygons are to each other as the squares of their corresponding sides.*



**Given** $a$ and $a'$ corresponding sides of two similar polygons whose areas are $S$ and $S'$ respectively.

**To prove** $\frac{S}{S'} = \frac{a^2}{a'^2}.$

### Proof

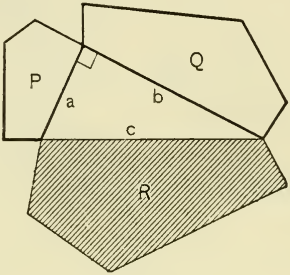
| STATEMENTS | REASONS |
|---|---|
| **1.** From two corresponding vertices draw all possible diagonals. | 1. Why? |
| **2.** $\triangle\text{I} \sim \triangle\text{I}'$, $\triangle\text{II} \sim \triangle\text{II}'$, etc. | 2. (315). |
| **3.** $\frac{\triangle\text{I}}{\triangle\text{I}'} = \frac{a^2}{a'^2}.$ | 3. (378). |
| **4.** $\frac{\triangle\text{I}}{\triangle\text{I}'} = \left(\frac{l^2}{l'^2}\right) = \frac{\triangle\text{II}}{\triangle\text{II}'} = \left(\frac{m^2}{m'^2}\right) = \frac{\triangle\text{III}}{\triangle\text{III}'}.$ | 4. (378). |
| **5.** $\frac{\triangle\text{I}}{\triangle\text{I}'} = \frac{\triangle\text{II}}{\triangle\text{II}'} = \frac{\triangle\text{III}}{\triangle\text{III}'}.$ | 5. Why? |
| **6.** $\frac{\triangle\text{I} + \triangle\text{II} + \triangle\text{III}}{\triangle\text{I}' + \triangle\text{II}' + \triangle\text{III}'} = \frac{\triangle\text{I}}{\triangle\text{I}'}.$ | 6. (284). |
| **7.** $S : S' = \triangle\text{I} : \triangle\text{I}'$ | 7. Why? |
| **8.** $\therefore\ S : S' = a^2 : a'^2.$ | 8. Why? |

**381.** COR. 1. *Similar polygons are to each other as the squares of their corresponding diagonals.*

**382.** COR. 2. *Similar polygons are to each other as the squares of their perimeters.*

## PROPOSITION XVI. THEOREM

**383.** *If similar polygons are constructed on the three sides of a right triangle, the sum of the polygons on the arms is equivalent to the polygon on the hypotenuse.*



**Given** the similar polygons $P$, $Q$, and $R$ constructed, respectively, on the arms $a$ and $b$ and the hypotenuse $c$ of the rt. $\triangle abc$.

**To prove** $P + Q = R$.

**Proof.** $\frac{P}{R} = \frac{a^2}{c^2}$ and $\frac{Q}{R} = \frac{b^2}{c^2}$. (380)

$\therefore \frac{P+Q}{R} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$. (Ax. 2, 329.)

*i.e.* $P + Q = R$. (Why?)

---

Ex. **1.** The areas of two similar polygons are respectively 3 sq. in. and 12 sq. in., and a side of the first is 6 in. Find the corresponding side of the second.

Ex. **2.** Two corresponding sides of two similar polygons are 4 in. and 3 in. respectively, the area of the greater polygon is 1 sq. ft. What is the area of the smaller polygon?

Ex. **3.** The area of a polygon is twice the area of a similar polygon. Find the ratio of any two corresponding sides.

Ex. **4.** On a map whose scale is 1 : 1000, how many square feet are represented by a polygon equivalent to 2 sq. in.?

Ex. **5**. Corresponding sides of 2 similar polygons have the ratio of 5 to 9, the sum of their areas is 212 sq. ft. Find the area of each figure.

Ex. **6**. Two similar parallelograms are to each other as the products of their diagonals.

Ex. **7**. Construct a triangle equivalent to two thirds of a given triangle, and similar to it.

Ex. **8**. Construct a triangle similar to a given triangle and three times as large.

Ex. **9**. Divide a triangle into two equivalent parts by a line parallel to one of its sides.

Ex. **10**. Divide a triangle into three equivalent parts by lines parallel to one of its sides.

Ex. **11**. Construct a polygon equivalent to two thirds of a given polygon and similar to it.

Ex. **12**. *Construct an equilateral triangle equivalent to the sum of two given equilateral triangles.*

Ex. **13**. *Construct an equilateral triangle equivalent to the difference of two given equilateral triangles.*

Ex. **14**. Construct an isosceles right triangle equivalent to the sum of two given isosceles right triangles.

Ex. **15**. Construct an isosceles right triangle equivalent to the difference of two isosceles right triangles.

Ex. **16**. *Construct a triangle similar to two given similar triangles and equivalent to their sum.*

Ex. **17**. Construct a polygon similar to two given similar polygons and equivalent to their sum.

Ex. **18**. Construct a polygon similar to two given similar polygons and equivalent to their difference.

Ex. **19**. Construct an equilateral triangle equivalent to the sum of three given equilateral triangles.

Ex. **20**. Two corresponding sides of two similar polygons are 12 feet and 5 feet respectively. Find the corresponding side of a similar polygon equivalent to their sum.

## MISCELLANEOUS EXERCISES

### THEOREMS

Ex. **1**. If $E$ is any point in the diagonal $AC$ of the parallelogram $ABCD$, prove that $\triangle AEB = \triangle ADE$.

Ex. **2**. A straight line passing through the intersections of the diagonals of a parallelogram divides the figure into two equivalent parts.

Ex. **3**. *The area of a circumscribed polygon is equivalent to one half the product of perimeter and radius.*

Ex. **4**. If through any point in a diagonal of a parallelogram parallels are drawn to the sides, four parallelograms are formed, of which the two which do not contain the diagonal are equivalent.

Ex. **5**. If any point within a parallelogram be joined to the four vertices, the sum of either pair of opposite triangles is equivalent to one half the parallelogram.

Ex. **6**. The lines joining the mid-points of the sides of a quadrilateral in succession form a parallelogram equivalent to one half the quadrilateral.

Ex. **7**. A rhombus and a square have equal perimeters. Compare their areas, given one angle of the rhombus is 60°.

Ex. **8**. If on the line joining the mid-points of two sides of a triangle a parallelogram is constructed having two of its vertices in the base of the triangle, the parallelogram is equivalent to one half the triangle.

Ex. **9**. The nonparallel sides of a trapezoid form with the diagonals two equivalent triangles.

Ex. **10**. If from the point of intersection of the medians of any triangle lines are drawn to the three vertices, they form with the sides three equivalent triangles.

Ex. **11**. If in the triangle $ABC$, $D$ and $F$ are the mid-points of the sides $AB$ and $AC$ respectively, the $\triangle ADC = \triangle ABF$.

Ex. **12**. *The area of an equilateral triangle whose side equals a is* $\frac{a^2}{4}\sqrt{3}$.

Ex. **13**. *The area of a rhombus is equal to half the product of the diagonals.*

## PROBLEMS OF COMPUTATION

Ex. **1**. The side of an equilateral triangle is 10 in. Find the area.

Ex. **2**. Find the area of an isosceles triangle if the base is 6 and an arm 5.

Ex. **3**. Find the area of a trapezoid whose bases are 9 and 11 respectively, and whose altitude is 12 ft.

Ex. **4**. Find the area of a rhombus whose diagonals are 9 and 10 ft. respectively.

Ex. **5**. Find the area of quadrilateral $ABCD$ if $AB = 10$, $BC = 24$, $CD = 30$, $AD = 28$, diagonal $AC = 26$.

Ex. **6**. A side of equilateral triangle $ABC$ is 8. Find the side of an equilateral triangle equivalent to three times triangle $ABC$.

Ex. **7**. The perimeter of a rectangle is 20 m., one side is 6 m. Find the area.

Ex. **8**. What is the side of a square whose area is 900 sq. m.? $n$ sq. m.?

Ex. **9**. The area of a rhombus is equal to $m$, and one diagonal is equal to $d$. Find the other diagonal.

Ex. **10**. The area of a trapezoid is 400 sq. m., its altitude is 8 m. Find the length of the line joining the mid-points of the nonparallel sides.

Ex. **11**. The hypotenuse of a right triangle is 20, and the projection of one arm upon the hypotenuse is 4. What is its area?

Ex. **12**. A farmer wishes to determine the area of a pentagonal field. He measures the lines $AB = 4$ rods, $BC = 13$ rods, $CD = 14$ rods, $DE = 5$ rods, $EA = 12$ rods, $AC = 15$ rods, and $AD = 13$ rods. How many square rods does the field contain?

Ex. **13**. The base and altitude of a triangle are 12 and 20 respectively. At a distance of 6 from the base, a parallel is drawn to the base. Find the areas of the two parts of the triangle.

Ex. **14**. Find the area of a rectangle having one side equal to 6 and a diagonal equal to 10.

Ex. **15**. A polygon with perimeter of 20 ft. is circumscribed about a circle whose radius is 3 ft. Find the area of the polygon.

Ex. **16**. Find the area of a triangle whose base is 10 inches and whose base angles are 120° and 30° respectively.

Ex. **17**. Find the side of an equilateral triangle equivalent to a rectangle, whose base and altitude are 10 and 15 respectively.

Ex. **18**. The chord of an arc is 42 in.; the chord of one half that arc is 29 in. Find the diameter of the circle.

Ex. **19**. The base of a triangle is 15 ft., its area is 60 sq. ft. Find the area of a similar triangle whose corresponding altitude is 6 ft.

Ex. **20**. An arm of an isosceles trapezoid is 13 in. and its projection on the longer base is 5 in.; the longer base is 17 in. Find the area of the trapezoid.

Ex. **21**. The arms of two isosceles right triangles are 3 and 2 respectively. Find the arm of an isosceles right triangle equivalent to their difference.

Ex. **22**. The sides of a triangle are as $8:15:17$. Find the sides if the area is 960 sq. ft.

Ex. **23**. The sides of a triangle are 8, 15, and 17. Find the radius of the inscribed circle.

HINT. Express in the form of an equation the fact that the area of the triangle is equal to the sum of the three triangles whose vertex is the incenter, and whose bases are the sides of the triangle.

Ex. **24**. Find the area of an equilateral polygon of 12 sides inscribed in a circle whose radius is 4 in.

Ex. **25**. The sides of a triangle are 6, 7, and 8 ft. Find the areas of the two parts into which the triangle is divided by the bisector of the angle included by 6 and 7.

Ex. **26**. Find the area of an equilateral triangle whose altitude is equal to $h$.

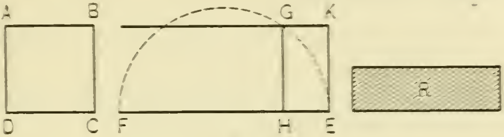
Ex. **27**. Find the area of a rhombus if the sum of its diagonals is 24 feet and their ratio is as $3 : 5$.

Ex. **28**. Upon the diagonal of a rectangle 16 ft. by 12 ft., a triangle equivalent to the rectangle is constructed. What is its altitude?

## PROBLEMS OF CONSTRUCTION

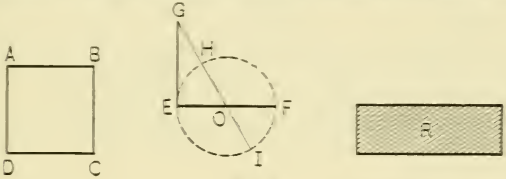
Ex. **1**. To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.

Hint. If $FE$ is the given line and $ABCD$ the given square, make $EK = AB$.



Ex. 1

Ex. **2**. To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.



Ex. 2

Hint. If $ABCD$ is the given square and $EF$ the given line, make $EG = AB$.

Ex. **3**. To transform a rectangle into another one, having a given side.

Ex. **4**. To transform a square into an isosceles triangle, having a given base.

Ex. **5**. To transform a rectangle into a parallelogram having a given diagonal.

Ex. **6**. To divide a triangle into three equivalent parts by lines parallel to a median.

Ex. **7**. To bisect a parallelogram by a line perpendicular to a side.

Ex. **8**. To divide a parallelogram into three equivalent parts by lines drawn through a vertex.

Ex. **9**. To bisect a trapezoid by a line drawn through a vertex.

Ex. **10**. Divide a pentagon into four equal parts by lines drawn through one of its vertices.

## MISCELLANEOUS

Ex. **1**. Show how to construct a line making an angle of 60° with a given line and tangent to a given circle.

Ex. **2**. One side of a rhombus is 20; the longer diagonal is 32. Find the other diagonal and the area.

Ex. **3**. Prove that the bisectors of the angles of a rectangle inclose a square.

Ex. **4**. Through a given point outside a given line to draw two lines, each making an angle of 45° with the given line.

Ex. **5**. A tower casts a shadow 120 feet long when a vertical 15-foot pole casts a shadow 10 feet long. How high is the tower?

Ex. **6**. Find the locus of the centers of all circles tangent to two concentric circles.

Ex. **7**. How far is a man from his starting point, if he first travels 5 miles north, then 12 miles east, and then 4 miles south?

Ex. **8**. Prove that the bisector of the right angle in any right triangle, if produced, passes through the center of the square having the hypotenuse as a side.

Ex. **9**. Construct a square equivalent to a given hexagon.

Ex. **10**. Two equilateral triangles have a side of one equal to the altitude of the other. Find the ratio of their areas.

Ex. **11**. Find the locus of the mid-points of all parallel chords in a circle.

Ex. **12**. On the base of a given triangle to construct an isosceles triangle equal in area to the given triangle.

Ex. **13**. Prove that the area of a triangle having an angle of 30° is equal to one fourth the product of the two sides including the angle of 30°.

Ex. **14**. Through a given point construct a line equidistant from two given points (2 sol.).

Ex. **15**. Find the side of an equilateral triangle whose area is $25\sqrt{3}$.

Ex. **16**. Divide a given arc of a circle into two parts which shall have their chords in a given ratio.

Ex. **17**. Two chords intersect within a circle; the segments of one are each 4, the total length of the other is 10. Find the length of its segments.

Ex. **18**. A house is 40 feet long, 30 feet wide, and 25 feet high to the roof and 35 feet to the ridgepole. Find the area of its exterior surface.

Ex. **19**. To construct a square which shall have a given ratio to a given square.

Ex. **20**. Prove: If the two diagonals of a quadrilateral divide each other in the same ratio, the quadrilateral is a trapezoid.

Ex. **21**. The radius of a circle is 13 inches. Through a point 5 inches from the center any chord is drawn. What is the product of the two segments of the chord?

Ex. **22**. The altitude upon the base of a triangle is 12. The two segments of the base made by the altitude are 9 and 16 respectively. Show that the triangle is a right triangle.

Ex. **23**. A side of a rhombus is 13 meters and one of its diagonals is 24 meters. Find its area.

Ex. **24**. $AB$ is any chord. $AC$ is a tangent and $CDE$ is a secant parallel to $AB$ and cutting the circle in $D$ and $E$. Prove $AC : AE = DC : BE$.

Ex. **25**. Divide a given line so that the product of its two segments equals a given square.

Ex. **26**. On one side of a triangle construct a rhombus equivalent to the given triangle.

# BOOK V

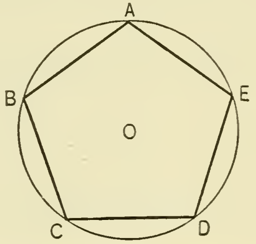
## REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE

### REGULAR POLYGONS

**384.** Def. A **regular polygon** is a polygon which is both equiangular and equilateral.

### Proposition I. Theorem

**385.** *An equilateral polygon inscribed in a circle is regular.*



**Given** circle O and inscribed equilateral polygon *ABCDE*.

**To prove** polygon *ABCDE* is regular.

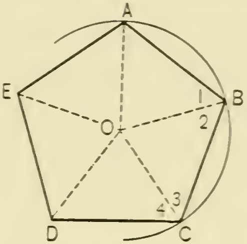
### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** $AB = BC = CD = DE \cdots$. | **1.** Hyp. |
| **2.** $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} \cdots$. | **2.** (187). |
| **3.** $\overset{\frown}{ABC} = \overset{\frown}{BCD} = \overset{\frown}{CDE} = \overset{\frown}{DEA} \cdots$ | **3.** Ax. 2. |
| **4.** $\therefore\ \angle B = \angle C = \angle D = \angle E = \angle A$. | **4.** Why? |
| **5.** The polygon is regular. | **5.** (384). |

[Practical applications, pages 297–300.]

### PROPOSITION II. THEOREM

**386.** *A circle can be circumscribed about any regular polygon.*



**Given** the regular polygon $ABCDE$.

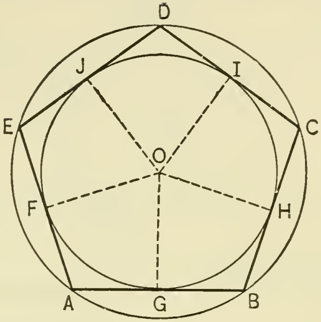
**To prove** that a circle can be circumscribed about $ABCDE$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. Construct a circle through $A$, $B$, and $C$ and let $O$ be its center. | 1. To pass a circle through three points not in a same st. line. |
| 2. Draw $OA$, $OB$, $OC$, $OD$. | 2. Why? |
| 3. $\angle ABC = \angle BCD$. | 3. Hyp. |
| 4. $OB = OC$. | 4. Why? |
| 5. $\angle 2 = \angle 3$. | 5. (71). |
| 6. $\angle 1 = \angle 4$. | 6. Why? |
| 7. $AB = CD$. | 7. Hyp. |
| 8. $\triangle OAB \cong \triangle OCD$. | 8. s. a. s. = s. a. s. |
| 9. $OA = OD$. | 9. Why? |
| 10. The ⊙ passes through $D$. | 10. Why? |
| 11. In like manner it can be proved that the circle passes through the other vertices of the polygon. | 11. Steps (1–10). |
| 12. A circle may be circumscribed about the given polygon. | 12. Why? |

### Proposition III. Theorem

**387.** *A circle can be inscribed in any regular polygon.*



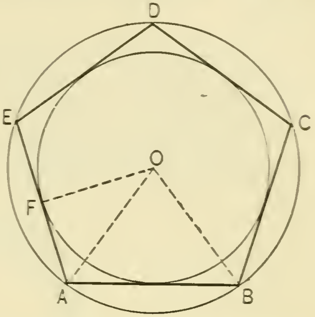
**Given** regular polygon $ABCDE$.

**To prove** a circle can be inscribed in the polygon.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Let a circle be circumscribed about the given regular polygon. | **1.** (386). |
| **2.** From center $O$ of circumscribed circle draw perpendiculars to each of the sides. | **2.** Why? |
| **3.** $AB = BC = CD = DE = EA$. | **3.** Hyp. |
| **4.** $OG = OH = OF$, etc. | **4.** (197). |
| **5.** With $O$ as center and with $OG$ as a radius describe a circle. | **5.** Why? |
| **6.** The circle passes through points $F, G, H, I, J$. | **6.** Why? |
| **7.** Sides $AB$, $BC$, $CD$, etc. are tangents to the circle having $OG$ as radius. | **7.** (203). |
| **8.** Hence a circle has been inscribed in the given regular polygon. | **8.** Each side of the polygon is a tangent to the ⊙. |

**388.** Def. The **center** (*O*) **of a regular polygon** is the common center of the circumscribed and inscribed circles of the polygon.



**389.** Def. The **radius of a regular polygon** is the radius of the circumscribed circle, as *OA*.

**390.** Def. The **central angle of a regular polygon** is the angle between two radii drawn to the ends of one side, as *AOB*.

**391.** Def. The **apothem of a regular polygon** is the radius of the inscribed circle, as *OF*.

**392.** Cor. *The central angle of a regular polygon of n sides is equal to* $\frac{4}{n}$ *right angles.*

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Ex. **1.** If *ABCDEF* is a regular hexagon, then

$$\angle ADC = \angle AEC = \angle AFC.$$

Ex. **2.** If two diagonals *AC* and *BE* of a regular polygon *ABCDEFG* intersect in *Q*, then $AQ \times QC = BQ \times QE$.

Ex. **3.** If in the regular heptagon *ABCDEFG*, the prolongations of *AB* and the diagonal of *EC* meet in *H*, then $HB \times HA = HC \times HE$.

Ex. **4.** *Any central angle of a regular polygon is the supplement of an angle of the polygon.*

Ex. **5.** A triangle is regular if its circumscribed and inscribed circles are concentric.

Ex. **6.** A polygon is regular if its circumscribed and inscribed circles are concentric.

Ex. **7.** How many degrees are there in a central angle of a regular triangle? In a regular octagon? In a regular hexagon?

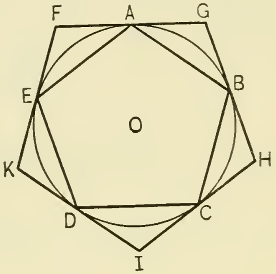
Ex. **8.** Find the apothem of a regular hexagon having each of its sides 6 inches.

### PROPOSITION IV. THEOREM

**393.** *If a circle is divided into any number of equal parts:*

(1) *The chords joining the successive points of division form a regular inscribed polygon;*

(2) *Tangents drawn at the points of division form a regular circumscribed polygon.*



**Given** the circle *ACE* divided into the equal arcs *AB*, *BC*, *CD*, etc., and chords *AB*, *BC*, *CD*, etc., and tangents *FG*, *GH*, *HI*, etc., at points of division.

(1) **To prove** *ABCDE* is a regular polygon.

**Proof.** **1.** $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$, etc. (Why?)

**2.** $AB = BC = CD, \cdots$ (Why?)

**3.** *ABCDE* is a regular polygon (385)

(2) **To prove** *FGHIK* is a regular polygon.

**Proof.** **1.** $\angle GAB = \angle GBA = \angle CBH = \angle HCB$, etc. (Why?)

**2.** $AB = BC = CD$, etc. (Why?)

**3.** $\triangle$ *ABG*, *CBH*, $\cdots$ are isos. and $\cong$ (Why?)

**4.** $\angle G = \angle H = \angle I$, etc. (Why?)

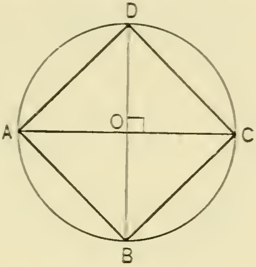
**5.** $AG = GB = BH = HC$, etc. (Why?)

**6.** $GH = HI = IK$, etc. (Ax. 2.)

**7.** $\therefore$ *FGHIK* is a regular polygon. (384)

### PROPOSITION V. PROBLEM

**394.** *To inscribe a square in a given circle.*



**Given** circle $O$.

**Required** to inscribe a square in the circle.

**Construction.** Draw any diameter $AC$. Construct the diameter $DB \perp AC$. Join $A$, $B$, $C$, $D$ in order. $ABCD$ is the required square.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** $DB \perp AC$. | **1.** Why? |
| **2.** $AD = DC = CB = BA$. | **2.** (125). |
| **3.** $ABCD$ is a regular quadrilateral; *i.e.* a square. | **3.** (385). |

**395.** COR. 1. *By bisecting the central angles, the arcs $AB$, $BC$, etc., will be bisected, and a regular octagon can be inscribed in the circle. By repeating the process, regular polygons of* 16, 32, $\cdots$, *sides can be constructed.*

**396.** COR. 2. *If $R$ is the radius of a circle, the side of the inscribed square equals $R\sqrt{2}$.*

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Ex. **1.** Circumscribe a square about a circle.

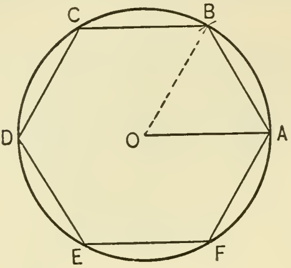
Ex. **2.** Circumscribe a regular octagon about a given circle.

Ex. **3.** Construct a regular octagon having a given side.

Ex. **4.** Find the area of a square, if its radius is equal to $r$

### PROPOSITION VI. PROBLEM

**397.** *To inscribe a regular hexagon in a given circle.*



**Given** circle $O$.

**Required** to inscribe a regular hexagon in the circle.

**Construction.** In the given circle $ACD$, draw the radius $AO$. From $A$ as a center, with a radius equal to $OA$, draw an arc meeting the circle in $B$. Draw $AB$. Then arc $AB$ will go exactly six times in the circle. Chord $AB$ goes six times also as a chord.

**Proof**

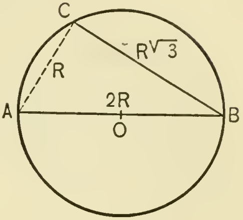
| STATEMENTS | REASONS |
|---|---|
| **1.** Draw $OB$. | **1.** Why? |
| **2.** $OA = OB = AB$. | **2.** Why? |
| **3.** $\angle O = \angle 60°$. | **3.** Why? |
| **4.** $\overset{\frown}{AB} = \overset{\frown}{60°}$. | **4.** Why? |
| **5.** $\overset{\frown}{AB}$ goes exactly 6 times in the $\odot$. | **5.** Why? |
| **6.** $\therefore AB = BC = CD$, etc. | **6.** (187). |
| **7.** $ABCDEF$ is a regular inscribed hexagon. | **7.** (385). |

**398.** COR. 1. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

**399.** COR. 2. *By joining the alternate vertices of an inscribed regular hexagon, an inscribed equilateral triangle is formed.*

**400.** Cor. 3. *Regular polygons of 3, 6, 12, 24, etc., sides can be inscribed in and circumscribed about a given circle.*

**401.** Cor. 4. *If R is the radius of a circle, the side of the inscribed equilateral triangle is* $R\sqrt{3}$.



Hint. If $AB$ is a diameter and $AC = R$, then $BC$ is the required side.

---

Ex. **1.** *If the radius of a circle is R, the apothem of the inscribed hexagon is equal to* $\frac{R}{2}\sqrt{3}$.

Ex. **2.** Circumscribe a regular hexagon about a given circle.

Ex. **3.** Construct a regular polygon of twelve sides, having given a side.

Ex. **4.** Find the area of a regular hexagon if its radius is equal to $R$.

Ex. **5.** The apothem of an equilateral triangle is equal to one half its radius.

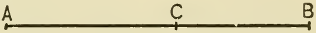
Ex. **6.** The area of an inscribed equilateral triangle is equal to one half the area of the inscribed regular hexagon.

* Ex. **7.** The areas of triangles inscribed in equal circles are to each other as the products of their three sides. (342)

Ex. **8.** A square constructed on a diameter of a circle is equivalent to twice the area of the inscribed square.

**402.** Def. A straight line is said to be divided in **extreme and mean ratio** when it is divided into two segments, such that *the greater is the mean proportional between the smaller and the whole line.*

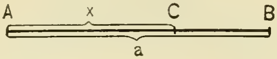
Thus, $AB$ is divided by $C$ in extreme and mean ratio, if



$$AB : AC = AC : CB.$$

### Proposition VII. Problem

**403.** *To divide a line in extreme and mean ratio.*



**Given** line $AB = a$.

**Required** to divide $a$ in extreme and mean ratio.

**Analysis.** Suppose $AC$, or $x$, is the greater of the required segments, then $CB = a - x$.

Hence, $a : x = x : a - x$.

$\therefore x^2 = a^2 - ax$.

Transposing, $x^2 + ax = a^2$.

Completing the square, $x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2$.

Extracting the root, $x + \frac{a}{2} = \sqrt{a^2 + \left(\frac{a}{2}\right)^2}$.

$$\therefore x = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2}.$$

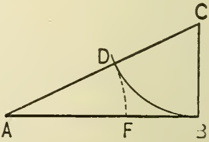
But since $\sqrt{a^2 + \left(\frac{a}{2}\right)^2}$ is the hypotenuse of a right triangle whose arms are $a$ and $\frac{a}{2}$, $x$ is easily constructed.

**Construction.** At $B$ draw $CB = \frac{a}{2}$ and $\perp AB$.

$\left[\text{Then } AC = \sqrt{a^2 + \left(\frac{a}{2}\right)^2}.\right]$

On $CA$ lay off $CD = CB \left(= \frac{a}{2}\right)$.

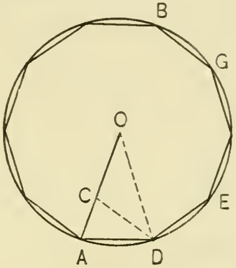
$\left[\text{Then } AD = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2} \text{ or } x.\right]$



On $AB$ lay off $AF = AD$. Then $AB$ is divided as required.

### PROPOSITION VIII. PROBLEM

**404.** *To construct the side of a regular decagon inscribed in a given circle.*



**Given** ⊙ $O$.

**Required** to construct the side of a regular inscribed decagon.

**Construction.** Divide a radius $OA$ in extreme and mean ratio. Draw chord $AD$ equal to $OC$, the greater segment of $OA$. $AD$ is a side of the required decagon.

**Proof.** Draw $DO$ and $DC$.

$$AO : OC = OC : CA, \qquad \text{(Con.)}$$

or since $OC = AD$,

$$AO : AD = AD : AC. \qquad \text{(Sub.)}$$

$$\therefore \triangle OAD \sim \triangle CAD. \qquad (311)$$

$$\therefore \angle O = \angle CDA. \qquad \text{(Why?)}$$

But since $\triangle OAD$ is isosceles, the similar triangle $DAC$ must be isosceles,

or, $CD = AD = CO$,

whence $\angle O = \angle CDO$. (Why?)

But $\angle O = \angle CDA$.

$$\therefore 2\angle O = \angle ADO = \angle A. \qquad \text{(Ax. 2.)}$$

But $\angle O + \angle D + \angle A = 2$ rt. $\angle$s.

$$\therefore 5\angle O = 2 \text{ rt. } \angle\text{s},$$

or $\angle O = \frac{2}{5}$ rt. $\angle$, or $\frac{1}{10}$ of 4 rt. $\angle$s.

$\therefore \overset{\frown}{AD}$ is $\frac{1}{10}$ of the circle, and $AD$ is a side of an inscribed regular decagon.

**405.** Cor. 1. *By joining the alternate vertices $A$, $E$, $G$,* etc., *a regular inscribed pentagon is formed.*

**406.** Cor. 2. *Regular polygons of 5, 10, 20, etc., sides can be inscribed in and circumscribed about a given circle.*

---

Ex. **1.** If $R$ is the radius of a circle, the side of the inscribed regular decagon equals $\frac{R}{2}(\sqrt{5} - 1)$.

Hint. Simplify the value of $x$ found in the analysis of Prop. VII. $(a = R.)$

Ex. **2.** Construct an angle of 36°.

Ex. **3.** Construct a regular decagon, having given a side.

Ex. **4.** Divide a right angle into five equal parts.

Ex. **5.** *The diagonals of a regular pentagon are equal.*

Ex. **6.** The diagonals of a regular pentagon divide each other in extreme and mean ratio.

Ex. **7.** The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Ex. **8.** *Any radius of a regular polygon bisects an angle of the polygon.*

Ex. **9.** The diagonals drawn from the vertex of a regular decagon divide that angle into eight equal parts.

Ex. **10.** Construct a regular pentagon, having given a side.

Ex. **11.** Explain a method by which we could construct an angle of 3°.

Ex. **12.** Prove: The square on a side of a regular inscribed pentagon equals the square of a radius and the square of a side of the regular inscribed decagon.

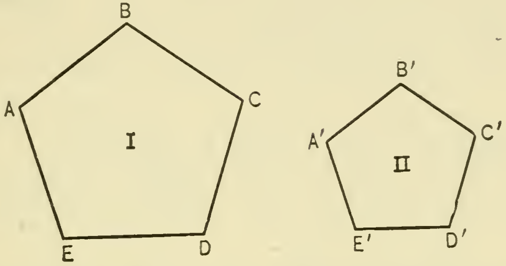
Ex. **13.** Find the side of a regular inscribed pentagon in terms of the radius ($R$) of its circumscribed circle.

Ex. **14.** Find the area of a regular pentagon if one side is equal to 4 inches.

Ex. **15.** Find the area of a regular decagon having side 4″.

### Proposition IX. Theorem

**407.** *Regular polygons of the same number of sides are similar.*



**Given** I and II, two regular polygons, each of $n$ sides.

**To prove** I ∼ II.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| 1. Each ∠ of I $= \frac{n-2}{n}$ st. ∠.<br>Each ∠ of II $= \frac{n-2}{n}$ st. ∠. | 1. (157). |
| 2. Each ∠ of I = each ∠ of II. | 2. Why? |
| 3. I and II are mutually equianglar. | 3. Why? |
| 4. $AB = BC = CD = DE = EA.$<br>$A'B' = B'C' = C'D' = D'E' = E'A'.$ | 4. (384). |
| 5. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$ | 5. Why? |
| 6. ∴ I ∼ II. | 6. (303). |

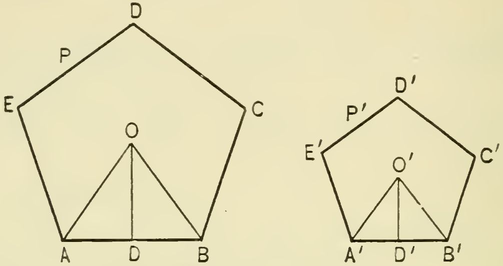
---

Ex. **1.** Construct a regular hexagon equivalent to one half a given regular hexagon.

Ex. **2.** Construct a regular pentagon equivalent to the sum of two given regular pentagons.

## PROPOSITION X. THEOREM

**408.** *The perimeters of regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.*



**Given** $P$ and $P'$, the perimeters of two regular polygons of the same number of sides, having the radii $OA$ and $O'A'$, and the apothems $OD$ and $O'D'$ respectively.

**To prove** $P : P' = OA : O'A' = OD : O'D'$.

### Proof

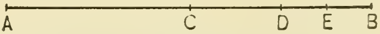
| | STATEMENTS | REASONS |
|---|---|---|
| 1. | $ABCDE \sim A'B'C'D'E'$. | 1. (407). |
| 2. | $P : P' = AB : A'B'$. | 2. (318). |
| 3. | $\angle AOB = \angle A'O'B'$. | 3. (392). |
| 4. | $AO = OB,\ A'O' = O'B'$. | 4. Why? |
| 5. | $\frac{AO}{A'O'} = \frac{OB}{O'B'}$. | 5. Why? |
| 6. | $\triangle AOB \sim \triangle A'O'B'$. | 6. (311). |
| 7. | $\frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OD}{O'D'}$. | 7. (303, 314). |
| 8. | $\therefore P : P' = OA : O'A' = OD : O'D'$. | 8. Why? |

**409.** COR. *The areas of regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

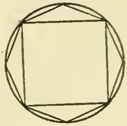
**410**. Def. A **constant** is a quantity that maintains the same value throughout a given discussion. A **variable** is a quantity whose value changes during the same discussion.

**411**. Def. If a variable $x$ approaches a constant $l$ so that the difference between $l$ and $x$ becomes and remains less than any assignable quantity different from zero, then $l$ is said to be a **limit** of $x$.

1. For example; suppose a point $p$ to take the positions $C$, $D$, $E$, etc., in succession, where $AC = \frac{1}{2} AB$, $CD = \frac{1}{2} CB$, $DE = \frac{1}{2} DB$, and so on, then the segments $AC$, $AD$, $AE$ approach $AB$ as a limit.



2. Inscribe a square in a circle, then inscribe an octagon by joining the mid-points of the four equal arcs to the vertices, and by continuing this process obtain regular inscribed polygons of 16, 32, 64, etc. sides. Obviously the area of the circle $C$ is larger than the area of any of these polygons $A$, but the difference between these two areas, $C-A$, becomes less if we increase the number of sides of the polygon. By continually increasing the number of sides of the polygon, the area of the polygon will approach the area of the circle more and more, or $C-A$ can be made less than any assignable quantity, however small.



The area of the circle is defined as the limit of the area of the polygon.

3. Similarly the perimeter of an inscribed polygon increases with the number of sides and approaches a limit. We define this limit to be the length of the circle (*i.e.* the circumference).

4. The sequence of fractions .9, .99, .999, ⋯, .9999,⋯, approaches a limit equal to 1, since we can, by going along far enough in the set of numbers, find that all that follow it, differ from 1 by as small a number as we please to choose.

**412**. The symbol for "approaches a limit" is represented by a horizontal arrow, $\longrightarrow$. "$x$ approaches $l$ as a limit" is represented by $x \longrightarrow l$.

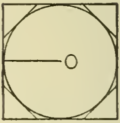
Note. In elementary geometry the variables that approach a limit as a rule cannot become equal to this limit. There are, however, cases in which variables become equal to their limits.

Ex. **1.** What is the limit of the fractions represented by $\frac{1}{n}$, if $n$ increases indefinitely?

Ex. **2.** What is the limit of the sum $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ ?

Ex. **3.** What is the limit of the areas of regular circumscribed polygons of $n$ sides if $n$ assumes the values 4, 8, 16, $\cdots$ to infinity?

Ex. **4.** What is the limit of the perimeters of regular circumscribed polygons of $n$ sides, if $n$ assumes the values 4, 8, 16, $\cdots$ to infinity?



Ex. 3

Ex. **5.** If the number of sides of regular inscribed polygons is increased indefinitely, what is the limit of (*a*) the apothems? (*b*) the sides?

Ex. **6.** If the number of sides of regular circumscribed polygons is increased indefinitely, what is the limit of the radii?

**413.** THEOREM. **If two variables, $x$ and $y$, are always equal and $x$ approaches $a$ as a limit, then $y$ approaches $a$ as a limit.**

Since $a - x = a - y$, and $a - x$ can be made less than any conceivable number,

$a - y$ can be made less than any conceivable number.

$\therefore y \longrightarrow a$.

**414.** COR. **If two variables are always equal, and each approaches a limit, their limits are equal.** (PRINCIPLE OF LIMITS.)

**415.** The following theorems are true:

(A) If $x \longrightarrow a$, then $nx \longrightarrow na$.

(B) If $x \longrightarrow a$, then $\frac{1}{n}x \longrightarrow \frac{1}{n}a$.

(C) If $x \longrightarrow a$, and $y \longrightarrow b$, then $(x \pm y) \longrightarrow (a \pm b)$.

(D) If $x \longrightarrow a$, and $y \longrightarrow b$, then $\frac{x}{y} \longrightarrow \frac{a}{b}$. $[b \neq 0]$.

(E) If $x \longrightarrow a$, and $y \longrightarrow b$, then $xy \longrightarrow ab$.

**Proof** of (A): $(a - x) \longrightarrow 0$, $n(a - x) \longrightarrow 0$, $(na - nx) \longrightarrow 0$. $\therefore nx \longrightarrow na$.

**416.** Def. The **length of a circle** (or, briefly, a circumference) is the limit of the perimeters of regular inscribed polygons, the number of whose sides is increased indefinitely.

Having defined a circumference, the meaning of the length of an arc is determined by § 228.

Note. It can be demonstrated that the limit of the perimeter of regular *circumscribed* polygons is the same as that of the inscribed ones. Hence a circumference can be defined as a common limit of the perimeters of regular inscribed and circumscribed polygons. For the purposes of this textbook, however, it is sufficient to consider the regular inscribed polygons only.

**417.** Teachers may desire to use the following *assumption* in place of the theory of limits. *Any property of a regular polygon which does not depend upon the number of sides of the polygon, is also a property of circles.*

**418.** Def. A **sector** of a circle is a figure formed by an arc and the two radii drawn to the ends of the arc.

**419.** Def. **Similar arcs**, **segments**, and **sectors** are those which correspond to equal central angles.

**420.** Def. The **area of a circle** is the limit of the area of regular circumscribed polygons whose sides are increased indefinitely.

Note. It can be proved that the areas of regular inscribed polygons approach the same limit as the areas of the circumscribed polygons. For the purposes of this book, however, it is more convenient to consider the circumscribed polygon only.

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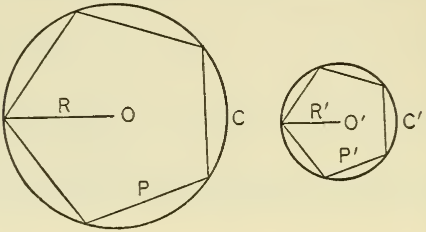
Ex. **1.** Prove theorems (B), (C), (D), and (E) in section 415.

Ex. **2.** Prove the lengths of similar arcs are in the same ratio as their radii.

Ex. **3.** Show that the areas of similar segments are in the same ratio as the squares of their radii.

## PROPOSITION XI. THEOREM

**421.** *Circumferences are to each other as their radii.*



**Given** circles $O$ and $O'$ in which $C$, $C'$ and $R$, $R'$ represent their circumferences and radii respectively.

**To prove** $C : C' = R : R'$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| 1. In each circle inscribe regular polygons of $n$ sides having perimeters $P$ and $P'$. | 1. Why? |
| 2. $P : P' = R : R'$. | 2. (408). |
| 3. $P : R = P' : R'$. | 3. (279). |
| 4. $P \longrightarrow C$ and $P' \longrightarrow C'$. | 4. (416). |
| 5. $\frac{P}{R} \longrightarrow \frac{C}{R}$, $\frac{P'}{R'} \longrightarrow \frac{C'}{R'}$. | 5. (415 B). |
| 6. $\therefore \frac{C}{R} = \frac{C'}{R'}$. | 6. (414). |
| 7. $\frac{C}{C'} = \frac{R}{R'}$, *i.e.* $C : C' = R : R'$. | 7. (279). |

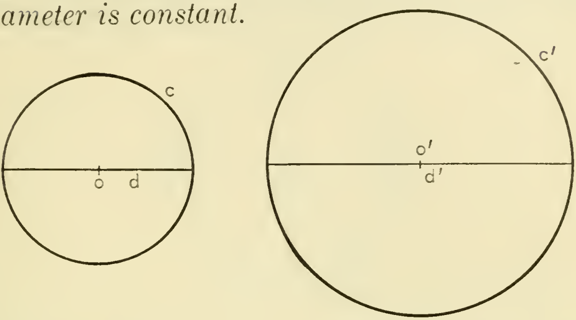
### Alternate Proof

| | |
|---|---|
| 1. Statement (1) above. | 1. Why? |
| 2. $P : P' = R : R'$. | 2. (408). |
| 3. $\therefore C : C' = R : R'$. | 3. (417). |

**422.** *Circumferences are to each other as their diameters.*

## PROPOSITION XII. THEOREM

**423.** *The ratio of the circumference of any circle to its diameter is constant.*



**Given** any circle $O$ with circumference $C$ and diameter $d$.

**To prove** that $C \div d =$ a constant.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Construct a fixed circle $O'$ with circum. $= C'$ and diameter $= d'$. | **1.** Why? |
| **2.** $C : C' = d : d'$. | **2.** (421). |
| **3.** $C : d = C' : d'$. | **3.** (279). |
| **4.** But $\frac{C'}{d'} =$ a constant. | **4.** Circle $O'$ is fixed. |
| **5.** $\therefore \frac{C}{d} =$ a constant. | **5.** Sub. |

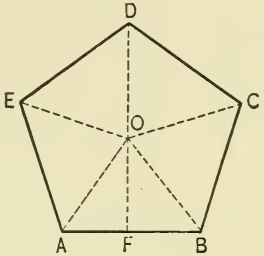
**424.** The constant $\frac{C}{d}$ is represented by the Greek letter $\pi$. Approximate values of $\pi$ are $3\frac{1}{7}$, 3.14, 3.1416.

Since $\frac{C}{d} = \pi$, then $C = \pi d = 2\pi R$.

**425.** *An arc of $m°$ in a circle of radius $R$ has its length equal to* $\frac{m}{360} \times 2\pi R$.

### PROPOSITION XIII. THEOREM

**426.** *The area of a regular polygon is equal to one half the product of its perimeter and apothem.*



**Given** $S$ the area, $P$ the perimeter, and $r$ the apothem of the polygon $ABCDE$, having $n$ sides.

**To prove** $S = \frac{1}{2} P \times r$.

**Proof**

| STATEMENTS | REASONS |
|---|---|
| **1.** Draw all the radii of the circumscribing circle: $OA$, $OB$, $OC$, etc., dividing the polygon into $n$ $\triangle$s with $O$ as vertex. | **1.** Why? |
| **2.** These $\triangle$s are all equal $\triangle$s. | **2.** s. s. s. = s. s. s. |
| **3.** Every triangle has its alt. equal to $r$. | **3.** (178). |
| **4.** $\triangle AOB = \frac{1}{2}(AB) \times r$. | **4.** (358). |
| **5.** $S = n \times \triangle AOB$. | **5.** Why? |
| **6.** $S = \frac{1}{2} \times n \times (AB) \times r$. | **6.** Sub. |
| **7.** But $n \times (AB) = P$. | **7.** Why? |
| **8.** $\therefore S = \frac{1}{2} P \times r$. | **8.** Sub. |

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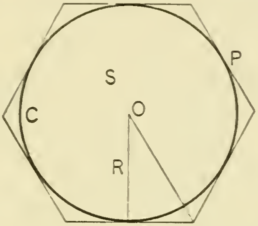
Ex. **1.** Find the area of a square circumscribed about a circle of radius 12″.

Ex. **2.** Find the area of a regular hexagon having a side of 6″.

### PROPOSITION XIV. THEOREM

**427.** *The area of a circle is equal to one half the product of its circumference and radius.*



**Given** $S$ the area of a circle of radius $R$ and circumference $C$.

**To prove** $S = \frac{1}{2} C \times R$.

### Proof

| STATEMENTS | REASONS |
|---|---|
| **1.** Circumscribe a regular polygon about the circle, and let $P$ be its perimeter, $A$ its area. | **1.** Why? |
| **2.** $A = \frac{1}{2} P \times R$. | **2.** (426). |
| **3.** Let the number of sides be increased indefinitely, then $A \longrightarrow S$. | **3.** (420). |
| **4.** But $P \longrightarrow C$. | **4.** (416). |
| **5.** $\frac{1}{2} P \times R \longrightarrow \frac{1}{2} C \times R$. | **5.** (415 A). |
| **6.** $\therefore S = \frac{1}{2} C \times R$. | **6.** (414). |

### Alternate Proof

| | |
|---|---|
| **1.** Same as (1) above. | **1.** Why? |
| **2.** $A = \frac{1}{2} P \times R$. | **2.** (426). |
| **3.** $S = \frac{1}{2} C \times R$. | **3.** (417). |

**428.** COR. 1. *The area of a circle is equal to $\pi$ times the square of its radius.*

For $S = \frac{1}{2} C \times R$ and $S = \frac{1}{2}(2\,\pi R)R = \pi R^2$.

**429.** Cor. 2. *The areas of circles are to each other as the squares of their radii, or of their diameters.*

**430.** Cor. 3. *The area of a sector whose central angle is $n°$ is equal to* $\frac{n}{360} \times \pi R^2$.

**431.** Cor. 4. *Similar sectors are to each other as the squares of their radii.*

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Ex. **1.** Construct a circle whose circumference shall equal the sum of the circumferences of two given circles.

Hint. Let $a$ and $b$ be the radii of the given circles and let $x$ equal the radius of the required circle.

Then $$2\pi x = 2\pi a + 2\pi b = 2\pi(a + b)$$
or $$x = a + b.$$

Ex. **2.** Construct a circle whose area shall equal the sum of the areas of two given circles.

Hint. $$\pi x^2 = \pi a^2 + \pi b^2 = \pi(a^2 + b^2)$$
or $$x^2 = a^2 + b^2.$$ See § 368, (8).

Ex. **3.** Construct a circle whose circumference equals the difference of the circumference of two given circles.

Ex. **4.** Construct a circle that has its area equal to the difference between the areas of two given circles.

Ex. **5.** Construct a circle having its area three times that of a given circle.

Ex. **6.** Construct a circle so that its circumference equals twice that of a given circle.

Ex. **7.** Construct a circle having its area equal to the area bounded by two given concentric circles.

Ex. **8.** Construct a circle whose circumference shall be equal to the semicircumference of a given circle.

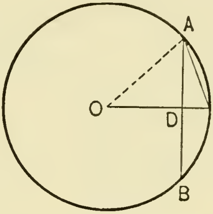
Ex. **9.** Find the difference of the areas of the circles circumscribed about and inscribed in a square.

Ex. **10.** What is the radius of a circle having an area of 616 sq. in.? ($\pi = \frac{22}{7}$.)

Ex. **11.** Find the area of a 32° sector in a circle of radius 7 in.

## PROPOSITION XV. PROBLEM

**432.** *The side and the radius of a regular polygon being given, to compute the side of a regular polygon of the same radius and double the number of sides.*



**Given** $AB$ equal to a side $s$ of a regular polygon inscribed in $\odot O$ with radius $OA = R$.

**Required.** To compute $AC$, the side of a regular polygon of double the number of sides, inscribed in the same circle.

**Construction.** Draw $CO$ meeting $AB$ in $D$. (Why?)

$OC$ bisects $AB$ at right angles. (79)

$$\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2\,OC \times OD, \quad 334)$$

or
$$\overline{AC}^2 = 2R^2 - 2R \times OD. \quad \text{(Sub.)}$$

But
$$\overline{OD}^2 = R^2 - \left(\frac{s}{2}\right)^2. \quad (330)$$

$$\therefore OD = \sqrt{R^2 - \frac{s^2}{4}},$$

or
$$OD = \tfrac{1}{2}\sqrt{4R^2 - s^2}.$$

Hence
$$\overline{AC}^2 = 2R^2 - R\sqrt{4R^2 - s^2}.$$

$$\therefore AC = \sqrt{2R^2 - R\sqrt{4R^2 - s^2}}.$$

**433.** COR. If $R = 1$, and the side of a polygon on $n$ sides equals $s_n$, then

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

## Proposition XVI. Problem

**434.** *To compute the ratio of the circumference of a circle to the diameter.*

Let $R = 1$, then $s_6 = 1$,
and according to (433), we have

| | Length of Side | Length of Perimeter |
|---|---|---|
| $s_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$ | $= 0.51764$ | 6.21166 |
| $s_{24} = \sqrt{2 - \sqrt{4 - (0.51764)^2}}$ | $= 0.26105$ | 6.26526 |
| $s_{48} = \sqrt{2 - \sqrt{4 - (0.26105)^2}}$ | $= 0.13081$ | 6.27870 |
| $s_{96} = \sqrt{2 - \sqrt{4 - (0.13081)^2}}$ | $= 0.06534$ | 6.28206 |
| $s_{192} = \sqrt{2 - \sqrt{4 - (0.06534)^2}}$ | $= 0.03272$ | 6.28291 |
| $s_{384} = \sqrt{2 - \sqrt{4 - (0.03272)^2}}$ | $= 0.01636$ | 6.28312 |
| $s_{768} = \sqrt{2 - \sqrt{4 - (0.01636)^2}}$ | $= 0.00818$ | 6.28317 |

Taking the last perimeter as the approximate value of the circumference,

$$\pi = \frac{C}{2R} = \frac{6.28317}{2} = 3.14159.$$

**435.** Method XXI. **Most computations referring to the circle involve the use of the two formulæ $C = 2\pi R$ and $S = \pi R^2$. If $R$ is not given, compute $R$ first.**

---

Ex. **1.** Find the circumference of a circle whose radius is 5.

Ex. **2.** The diameter of a circle equals 10 in. Find the length of an arc of (*a*) 60°, (*b*) 1°, (*c*) 5°.

Ex. **3.** Find the area of a circle whose radius is 10.

Ex. **4.** Find the radius of a circle whose area is 100 sq. in.

Ex. **5.** Find the radius of a circle whose circumference equals $m$ inches.

Ex. **6.** Find the circumference of a circle whose area equals $100\pi$.

Ex. **7**. Find the area of a circle whose circumference equals 4.

Ex. **8**. Find the circumference of a circle whose area equals $S$.

Ex. **9**. Find the area of a circle whose circumference equals $C$.

Ex. **10**. Find the radius of a circle equivalent to a square whose side equals 6.

Ex. **11**. The circumference of a circle equals 10. Find the circumference of a circle having twice the area of the given circle.

Ex. **12**. Two concentric circles have their circumferences equal to 30 and 40 respectively. Find the area bounded by the two circles.

Ex. **13**. Find a semicircle equivalent to an equilateral triangle whose side equals 5.

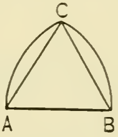
Ex. **14**. Find the area of a sector whose radius equals 5, and whose central angle equals 40°.

Ex. **15**. A square is inscribed in a circle of radius 10. Find the area of a segment having a side of the square as its chord.

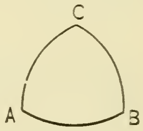
Ex. **16**. The radius of a circle is 4. What is the area of a segment whose arc is ($a$) 120°, ($b$) 60°?

Ex. **17**. In the diagram given here, $AB = BC = CA = 4$, $B$ is the center of arc $AC$, and $A$ is the center of arc $BC$. Find the total area.

Ex. **18**. $A$ is the center of arc $BC$, $B$ the center of $AC$, and $C$ the center of $AB$. Find the total area, if $AB = 6$ in.



Ex. 17



Ex. 18

Ex. **19**. A circle has an area of 60 sq. ft. Find the length of an arc of 60°.

Ex. **20**. Show how to inscribe a circle in a given sector.

Ex. **21**. The radius of a circle is 20 ft.; the area of a sector of that circle is $25\pi$ sq. ft. What is its arc in degrees?

Ex. **22**. The area of a sector is $18\pi$; the angle of the sector is 80°. Find the radius of the sector.

Ex. **23**. The altitude of an equilateral triangle is 6 in. Find the area of the circumscribed and inscribed circles.

Ex. **24**. The areas of two circles are as $9:4$; the radius of the larger is 6 in. Find the circumference of the smaller circle.

Ex. **25**. Find the area of a circle inscribed in a square whose area is $12\frac{1}{4}$ sq. in.

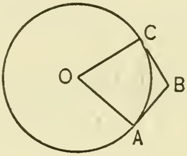
Ex. **26**. If the apothem of a regular hexagon is 4, what is the area of its circumscribed circle?

Ex. **27**. It requires $41\frac{1}{7}$ rd. of fencing to inclose a semicircular field. Find the area of the field. (Assume $\pi = 3\frac{1}{7}$.)

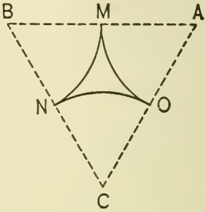
Ex. **28**. Prove that the star-shaped polygon formed by producing the sides of a regular hexagon is equivalent to twice the hexagon.

Ex. **29**. If $AB$ and $BC$ are tangents, $\angle COA = 60°$, and $OA = 10$ in. Find (*a*) area $OABC$, (*b*) area bounded by arc $AC$ and the two tangents.

Ex. **30**. Find the area bounded by the arcs $MN$, $NO$, and $OM$, if each arc $= 60°$ and their common radius equals 10.



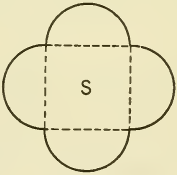
Ex. 29



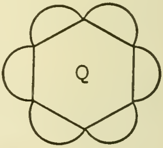
Ex. 30

Ex. **31**. On the sides of a square whose side is 4 in., semicircles are constructed. Find the total area.

Ex. **32**. On the sides of a regular hexagon whose side equals 2 in., semicircles are drawn. Find the total area.



Ex. 31



Ex. 32

Ex. **33**. Find the radius of a circle whose circumference numerically equals its area.

Ex. **34**. Find the central angle of a sector whose area is equal to the square of the radius.

Ex. **35**. The area of a square inscribed in a circle is 16 sq. in. Find the area of the circle.

Ex. **36**. Find the diameter of a wheel which, in going a mile, makes 500 revolutions.

## MISCELLANEOUS EXERCISES

Ex. **1**. Find the radius, the apothem, and the area of the following *regular* polygons:

(*a*) A square whose side equals 4
(*b*) A square whose side equals $s$
(*c*) A hexagon whose side equals 2
(*d*) A hexagon whose side equals $s$
(*e*) A triangle whose side equals 6
(*f*) A triangle whose side equals $s$

Ex. **2**. A side of a regular circumscribed triangle is equal to twice a side of a regular inscribed triangle.

Ex. **3**. The square of a side of an inscribed equilateral triangle is three times the square of a side of the regular inscribed hexagon.

Ex. **4**. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

Ex. **5**. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

Ex. **6**. The area of an equilateral triangle is $36\sqrt{3}$ units. Find the area of the inscribed circle.

Ex. **7**. Prove that an exterior angle of a regular polygon made by producing one side is equal to the angle at the center of the polygon.

Ex. **8**. A radius of a circle is 12 in. Find the area and the perimeter of a segment of this circle whose chord equals the radius.

Ex. **9**. Find the number of square inches of tin that would be wasted in cutting the largest possible circular disk from a piece in the form of an equilateral triangle 12 in. on a side.

Ex. **10**. The side of a square is 12. Find the area included between the square and the circumscribed circle.

Ex. **11**. A village has two standpipes each 50 feet high, one 10 feet in diameter and the other 16 feet in diameter. These two standpipes are to be replaced by one of the same height and capacity. Find the diameter of the new standpipe.

## GENERAL REVIEW EXERCISES

Ex. **1**. Prove that the medians drawn to the legs of an isosceles triangle are equal.

Ex. **2**. The triangle $ABC$ has a right angle at $C$ and $D$ trisects $AC$, so that $AD = 2\,DC$. It is found that $AD = BD$. Find the size of $\angle A$.

Ex. **3**. Prove that two lines drawn from the extremities of a side of a triangle and limited by the other sides cannot bisect each other.

Ex. **4**. Show how to divide a right triangle into two isosceles triangles.

Ex. **5**. In a regular polygon of 15 sides, find the number of degrees in (1) each interior angle, in (2) each exterior angle.

Ex. **6**. In a quadrilateral the diagonals are equal and also one pair of opposite sides are equal. Prove that two of the triangles into which the quadrilateral is divided by the diagonals are isosceles.

Ex. **7**. The median of a trapezoid bisects the two diagonals.

Ex. **8**. If a diagonal of a parallelogram bisects one angle, the figure is a rhombus.

Ex. **9**. Draw a line terminated by the sides of a given angle, so that it shall be equal to a given line and parallel to another given line.

Ex. **10**. Draw a circle concentric with a given circle and having a given chord in the given circle as a tangent.

Ex. **11**. What is the locus of the vertex of a 60° angle the sides of which always pass through two fixed points?

Ex. **12**. Three consecutive sides of an inscribed quadrilateral subtend arcs of 70°, 85°, and 98° respectively. Find each angle of the quadrilateral, an angle between its diagonals, and the angles formed by producing the opposite sides until they meet.

Ex. **13**. Prove that if a trapezoid be inscribed in a circle, the nonparallel sides produced will meet on the perpendicular bisector of the parallel sides.

Ex. **14**. Find the locus of the centers of all circles of a given radius that cut off a given distance on a given line.

Ex. **15**. Upon a given line construct accurately with ruler and compasses a segment of a circle that contains an angle of 30°.

Ex. **16**. Given two diagonals of a regular pentagon intersecting within it. Find the number of degrees in an angle between them.

Ex. **17**. Through a point $P$ within a circle a variable chord $APB$ is drawn meeting the circle in $A$ and $B$. Find the locus of the middle point of $AB$.

Ex. **18**. Show how to draw to a given circle two tangents which shall form a given angle.

Ex. **19**. If two polygons are similar to a third polygon, they are similar to each other.

Ex. **20**. From a point without a circle two tangents are drawn making with each other an angle of 60°. The length of each tangent is 15 inches. Find the diameter of the circle.

Ex. **21**. Two chords intersect within a circle; the segments of one are each 6, the total length of the other is 13. Find the length of its segments.

Ex. **22**. A rod 8 feet long is free to move within a rectangle 8 feet long and 6 feet wide. Describe accurately the boundary of the region within which the middle point of the rod will be found.

Ex. **23**. Construct an equilateral triangle given: (1) the radius of its inscribed circle; (2) the radius of its circumscribed circle.

Ex. **24**. The chord of a segment of a circle is 34 in. in length, and the radius of the circle is 60 in. Find the height of the segment.

Ex. **25**. The hypotenuse of a right triangle is 10 inches long and one angle is 30°. Find the lengths of the segments into which the shorter leg is divided by the bisector of the opposite angle.

Ex. **26**. The altitude of a triangle is 12 feet, and it divides the base into segments that are 5 feet and 9 feet. Find the perimeter of the triangle.

Ex. **27**. Prove that if two equivalent triangles have two sides in each respectively equal then the included angles are either equal or supplementary.

Ex. **28**. From a given point on a circle draw the chords that are bisected by a given chord. When is this impossible?

Ex. **29**. The radii of two circles are 7 inches and 2 inches, and their centers are 13 inches apart. Find the length of their common tangents.

Ex. **30**. Prove that if a point move about within a regular polygon, the sum of the perpendiculars let fall upon the sides (or the sides produced) will be constant.

Ex. **31**. The bases of a trapezoid are 29 feet and 37 feet respectively, and its area is 247.5 square feet. Find its altitude.

Ex. **32**. The sides of a triangle are 5 5, and 6. Find (1) its area, (2) the area of its inscribed circle, (3) the altitude upon one of the sides 5, (4) the diameter of its circumscribing circle.

Ex. **33**. The hypotenuse of an isosceles right triangle is 10 inches. What is the length of a side of the square whose area is three times that of the triangle?

Ex. **34**. Given an irregular hexagon to construct an equivalent square.

Ex. **35**. One diagonal of a rhombus is $\frac{8}{15}$ of the other, and the difference of the diagonals is 14. Find the area and the perimeter of the rhombus.

Ex. **36**. In a circle of area 275 square feet there is inscribed a rectangle of 150 square feet. Find the lengths of the sides of the rectangle.

Ex. **37**. The base of a triangle is 20 and its altitude is 30. Find the area of the trapezoid cut off by a line parallel to the base at a distance of 10 from it.

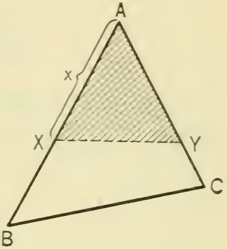
Ex. **38**. What equal regular polygons will exactly cover a plane surface when they are of the same number of sides? when any combination is allowed?

Ex. **39**. Express the area of a regular hexagon in terms of one of its sides ($a$).

## SOLUTION OF PROBLEMS BY MEANS OF ALGEBRAIC ANALYSIS

**436.** If a problem requires the construction of lines, it is often possible to state the condition in the form of an equation. The solution of the equation gives the unknown line in an algebraic form, which may be constructed.

Ex. A. From a triangle $ABC$, to cut off an isosceles triangle $AXY$, equivalent to one half of $ABC$.



**Analysis.** Let

$$AX = AY = x,$$
$$AB = c,$$

and

$$AC = b.$$
$$\triangle AXY : \triangle ABC = x^2 : bc = 1 : 2. \text{ (Why?)}$$
$$\therefore 2x^2 = bc.$$
$$x = \sqrt{\frac{bc}{2}}.$$

**Construction.** Construct $x$, a mean proportional between $\frac{c}{2}$ and $b$, and on $AB$ and $AC$ respectively, lay off $AX$ and $AY$ equal to $x$.

Ex. B. From a point outside a circle, to draw a secant which is bisected by the circle.

**Analysis.** Let $PAB$ be any secant. Suppose $PCD$ is the required secant.

Then $PC = CD$ or $PD = 2\,PC$.

Designate $PB$ and $PA$ by known letters $b$ and $a$.

Then, by § 322,

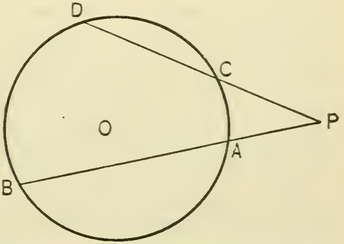
$$PC \cdot PD = PB \cdot PA$$

or, putting $PC = x$,

$$x \cdot 2x = b \cdot a,$$
$$2x^2 = ba,$$
$$x^2 = \frac{ba}{2} = \frac{b}{2} \cdot a.$$

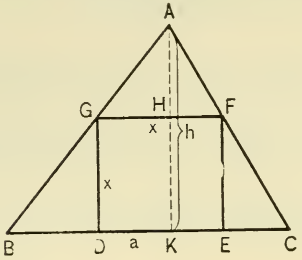
This becomes

$$\frac{b}{2} : x = x : a.$$



**Construction.** Construct $x$, a mean proportional between $\frac{b}{2}$ and $a$.

Ex. C. To inscribe a square in a given triangle.



**Given** $\triangle ABC$ to inscribe a square in this $\triangle$.

**Analysis.** Let $x$ = a side of the required square.

Let the altitude $AK = h$

and the base $BC = a.$

By similar $\triangle$s, $BC : GF = AK : AH,$

or $a : x = h : h - x.$

Then $xh = ah - ax,$

and $x = \dfrac{ah}{a + h} = \dfrac{a \cdot h}{(a + h)}.$

In the form of a proportion, $a + h : a = h : x.$

**Construction.** Construct $x$ as the fourth proportional to $a + h$, $a$, and $h$.

---

Ex. **1.** To divide the longer side of a rectangle into two parts so that the difference of their squares equals the area of the rectangle.

Ex. **2.** In a given square, to inscribe another square, having a given side.

Ex. **3.** To inscribe a square in a semicircle.

Ex. **4.** To transform a given square into a rectangle having a perimeter equal to twice the perimeter of the given square.

Ex. **5.** Given two concentric circles, to draw a chord in the larger circle so that it equals twice the chord formed in the smaller circle.

Ex. **6.** In the line $AB$, to find a point $C$ so that $\overline{AC}^2 = 3\,\overline{CB}^2$.

Ex. **7.** To construct a square equivalent to the difference between a given square diminished by one half a second given square.

Ex. **8.** Through a given point within a circle, to draw a chord so that the given point is a trisection point of the required chord.

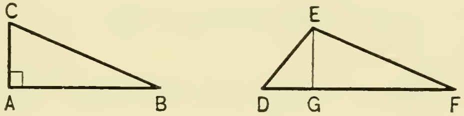
## MAXIMA AND MINIMA OF PLANE FIGURES

**437.** Def. The greatest of all magnitudes that satisfy given conditions is called the **maximum**; the least is called the **minimum.**

**438.** Def. **Isoperimetric figures** are those which have **equal perimeters.**

### Proposition I. Theorem

**439.** *Of all triangles having given two sides, that in which those two sides are perpendicular to each other is a maximum.*



**Given** in $\triangle$ $ABC$ and $DFE$, $AB = DF$, $AC = DE$, $\angle A =$ rt. $\angle$, and $\angle D$ oblique.

**To prove** area $ABC >$ area $DEF$.

**Proof.** From $E$ draw $EG \perp DF$.

Then $EG < DE$, (Why?)

or $EG < AC$.

$\therefore$ $\triangle$ $ABC$ and $DEF$ have equal bases and unequal altitudes.

$\therefore$ $\triangle ABC > \triangle DEF$. (Why?)

$\therefore$ $\triangle ABC$ is a maximum.

---

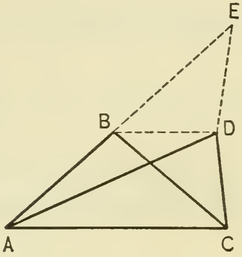
Ex. **1.** Of all triangles having given the base and the median to the base, which is the maximum?

Ex. **2.** Of all parallelograms having given the diagonals, which has the maximum area?

Ex. **3.** To divide a given line into two parts such that the rectangle contained by the segments is a maximum.

### PROPOSITION II. THEOREM

**440.** *Of all triangles having the same base and equal areas, the isosceles triangle has the minimum perimeter.*



**Given** $\triangle ADC = \triangle ABC$, and $AB = BC$.

**To prove** $AB + BC + AC < AD + DC + AC$.

**Proof.** Produce $AB$ by its own length to $E$. Draw $BD$ and $DE$.

$$BD \parallel AC$$

(*for otherwise* $\triangle ABC$ *would not be* $= \triangle ADC$).

$\angle EBD = \angle BAC.$ (Why?)

$\angle BAC = \angle BCA.$ (Why?)

$\angle BCA = \angle CBD.$ (Why?)

$\therefore \angle EBD = \angle CBD.$ (Ax. 1.)

But $BC = BE$, (Why?)

and $BD$ is common.

$\therefore \triangle BDC \cong \triangle BDE.$

$\therefore DC = DE.$

But $AD + DE > AB + BE.$ (Why?)

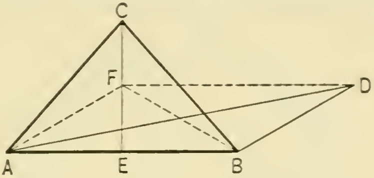
$\therefore AD + DC > AB + BC$,

or adding $AC$,

$$AC + AD + DC > AB + BC + AC.$$

### PROPOSITION III. THEOREM

**441.** *The maximum of isoperimetric triangles on the same base is the one whose other two sides are equal.*



**Given** $\triangle$ $ABC$ and $ABD$ having equal perimeters, the common base $AB$, and $AC = CB$.

**To prove** area $ACB >$ area $ADB$.

**Proof.** Draw median $CE$, and $DF \parallel AB$, meeting $CE$ in $F$. Draw $FA$ and $FB$.

Then $CE$ is the perpendicular bisector of $AB$. (Why?)

$$\therefore AF = FB.$$

But $\triangle AFB = \triangle ADB$. (359)

$\therefore$ perimeter $AFB <$ perimeter $ADB$. (440)

$$\therefore AF + FB < AD + DB.$$

But $AB + DB + DA = AB + BC + CA$. (Hyp.)

$\therefore DB + DA = BC + CA$. (Ax. 3.)

$\therefore AF + FB < BC + CA$. (Sub.)

But $AF = FB$ and $BC = CA$.

Hence $AF < AC$.

$\therefore FE < CE$. (330)

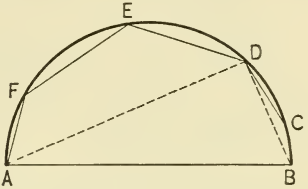
$\therefore$ area $AFB <$ area $ACB$, (361)

or area $ADB <$ area $ACB$.

**442.** COR. *Of all isoperimetric triangles, the equilateral has the maximum area.*

## PROPOSITION IV. THEOREM

**443.** *Of all polygons having all sides given in order but one, the polygon having the maximum area can be inscribed in a semicircle having the undetermined side as diameter.*



**Given** polygon *ABCDEF*, the maximum of all polygons, having given the sides *AF*, *FE*, *ED*, *DC*, and *CB*.

**To prove** *ABCDEF* can be inscribed in a semicircle whose diameter is *AB*.

**Proof.** Join any vertex, as *D*, with *A* and *B*.

Then $\triangle ADB$ must be the maximum of all triangles that can be formed with sides *AD* and *DB*; *i.e.* $\angle ADB$ must be a rt. $\angle$. For otherwise by making $\angle ADB$ a right one without changing the sides *AD* and *DB*, we could increase $\triangle ADB$ without altering the remaining parts *AFED* and *DCB* of the polygon. Or polygon *ABCDEF* would be increased, which is contrary to the hypothesis, since *ABCDEF* is a maximum.

$$\therefore \angle ADB \text{ is a right angle,}$$

and *D* is on a semicircumference that can be constructed on *AB*.

For the same reason every vertex of the polygon must lie on the semicircumference.

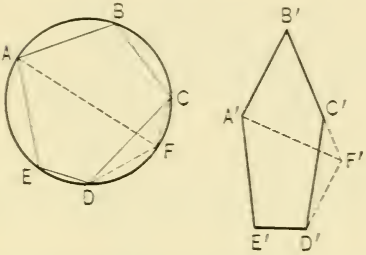
NOTE. We assume (1) a maximum, and (2) but one maximum.

Ex. **1.** In quadrilateral *ABCD*, $AB = BC = CD =$ a given line. Determine the angles *A*, *B*, *C*, and *D* so that *ABCD* becomes a maximum.

Ex. **2.** To inscribe an angle in a given semicircle so that the sum of its arms is a maximum.

### PROPOSITION V. THEOREM

**444.** *Of all polygons constructed with the same given sides in a given order, that which can be inscribed in a circle has the maximum area.*



**Given** polygon $ABCDE$ inscribed in a circle, mutually equilateral with polygon $A'B'C'D'E'$, which cannot be inscribed in a circle.

**To prove** area $ABCDE >$ area $A'B'C'D'E'$.

**Proof.** From $A$ draw diameter $AF$, and join $F$ to the two nearest vertices $C$ and $D$.

On $C'D'$, construct $\triangle C'D'F'$ congruent to $\triangle CDF$, and join $A'F'$.

Area $ABCF >$ area $A'B'C'F'$. (443)

Area $FDEA >$ area $F'D'E'A'$. (443)

$\therefore$ area $ABCFDE >$ area $A'B'C'F'D'E'$.

But $\triangle CFD \cong \triangle C'F'D'$.

Then area $ABCDE >$ area $A'B'C'D'E'$. (Ax. 5.)

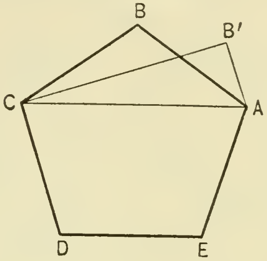
---

Ex. **1.** In quadrilateral $ABCD$, $AB = 2$ inches, $BC = CD = DA = 1$ inch. What must be the angles $A$, $B$, $C$, and $D$ to make the area $ABCD$ a maximum?

Ex. **2.** Determine the angles of pentagon $ABCDE$, if its area is a maximum and $AB = BC = CD = 2$ inches, and $DE = EA = 2\sqrt{2}$ inches.

### Proposition VI. Theorem

**445.** *Of all isoperimetric polygons of the same number of sides, the equilateral has the maximum area.*



**Given** *ABCDE* the maximum of all isoperimetric polygons of the same number of sides.

**To prove** $AB = BC = CD = DE = EA$.

**Proof.** Suppose polygon $AB'CDE$, containing the two unequal sides $CB'$ and $AB'$, were the maximum.

Then construct on the diagonal $AC$ the isosceles triangle $AB'C$, isoperimetric with $ABC$. (441)

Area $AB'C$ is less than area $ABC$.

∴ polygon $AB'CDE$ is less than the isoperimetric polygon $ABCDE$.

Hence polygon $AB'CDE$ cannot be the maximum.

Or in the maximum polygon $AB = BC$. Therefore all sides of the maximum polygon are equal.

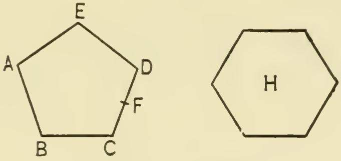
**446.** Cor. *Of all isoperimetric polygons of the same number of sides,* **the maximum is regular.** (444, 445)

---

Ex. In a given semicircle to inscribe a trapezoid whose area is a maximum.

## PROPOSITION VII. THEOREM

**447.** *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



**Given** regular pentagon *ABCDE* isoperimetric with regular hexagon *H*.

**To prove** area *H* > area *ABCDE*.

**Proof.** Let *F* be any point in *CD*.

*ABCFDE* may be considered a hexagon, having one of its angles equal to a straight angle.

*ABCFDE* is not an equilateral polygon. *H* is an equilateral polygon.

$\therefore$ area *H* > area *ABCDE*. (445)

**448.** COR. *The area of a circle is greater than the area of any polygon whose perimeter equals the circumference of the circle.*

---

Ex. **1**. In a given segment to inscribe an angle so that the sum of its arms is a maximum.

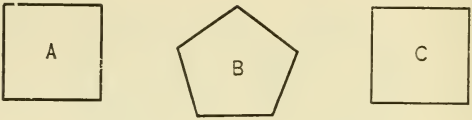
Ex. **2**. In a given circle to inscribe a triangle, having a maximum perimeter.

Ex. **3**. The circumscribed regular polygon has a smaller area than any other polygon circumscribed about the same circle.

Ex. **4**. Of all equivalent parallelograms on the same base, which has the minimum perimeter?

## Proposition VIII. Theorem

**449.** *Of two equivalent regular polygons, that which has the greater number of sides has the smaller perimeter.*



**Given** square $A$ = regular pentagon $B$.

**To prove** perimeter of $A >$ perimeter of $B$.

**Proof.** Construct square $C$ isoperimetric with $B$.

Area $C <$ area $B$, (447)

or area $C <$ area $A$.

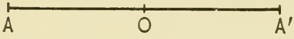
$\therefore$ perimeter of $C <$ perimeter of $A$.

$\therefore$ perimeter of $B <$ perimeter of $A$.

**450.** Cor. *The circumference of a circle is less than the perimeter of any equivalent polygon.*

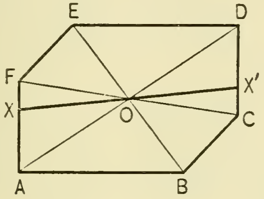
## SYMMETRY

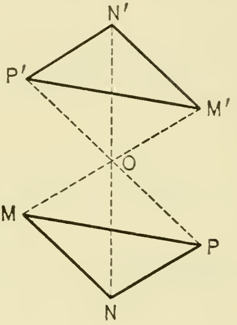
**451**. Two points $A$ and $A'$ are **symmetric with respect to a point** $O$, if $O$ is the mid-point of $AA'$.



**452**. A figure is **symmetric with respect to a point** $O$, if any point in the figure is symmetric to another point in the figure with respect to $O$.

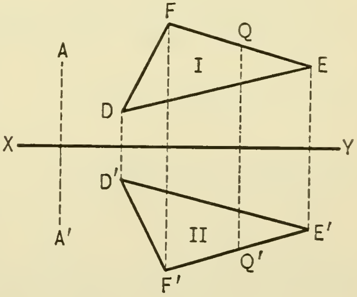
Thus, $ABCDEF$ is a figure symmetric to $O$. Obviously any line drawn through $O$ and terminated by the figure, as $XOX'$, is bisected. If the figure consists of two parts, as $\triangle MNP$ and $\triangle M'N'P'$, the two parts (*i.e.* the two triangles) are symmetric with regard to $O$.

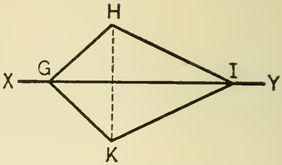




**453**. Two points $A$ and $A'$ are **symmetric with regard to an axis** $XY$, if $XY$ is the perpendicular-bisector of $AA'$.

**454**. Two figures are **symmetric with regard to an axis** $XY$, if to every point in the figure $Q$, there exists a corresponding point in the other, $Q'$, so that $Q$ and $Q'$ are symmetric with respect to $XY$.



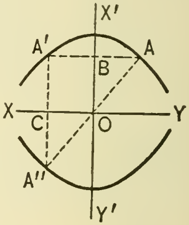


Thus, $\triangle DEF$ and $\triangle D'E'F'$ are symmetric with respect to $XY$. The two figures may also form parts of one figure, as $GHIK$. Then the entire figure $GHIK$ is said to be symmetric with respect to $XY$.

**455.** If two figures I and II are symmetric with regard to an axis $XY$ and one figure I is turned about $XY$ until it lies on the same side of $XY$ as the other figure II, then I and II will coincide.

Conversely, if such a rotation produces a coincidence of the figures, they are symmetric with respect to $XY$.

**456.** THEOREM. *If a figure is symmetric with respect to two axes perpendicular to each other, it is symmetric with respect to their point of intersection.*



Let $A$ be a point in the figure. There must exist another point $A'$ so that $AA' \perp X'Y'$ and $AB = A'B$. Similarly there must be another point $A''$ so that $A'A'' \perp XY$ and $A'C = A''C$. Obviously $AA' \parallel XY$ and $A'A'' \parallel XY$.

Draw $AA''$.

Then $X'Y'$ bisects $AA''$. (148)

Similarly $XY$ bisects $AA''$. (148)

$\therefore$ $XY$ and $X'Y'$ meet $AA''$ in its mid-point.

Or line $AA''$ is bisected by $O$, the point of intersection of $XY$ and $X'Y'$. Or to any point $(A)$ there exists another $A''$ so that $A$ and $A''$ are symmetric with regard to $O$.

---

Ex. **1**. A parallelogram is symmetric with respect to the point of intersection of its diagonals.

Ex. **2**. An isosceles triangle is symmetric with respect to the median drawn to the base.

Ex. **3**. Quadrilateral $ABCD$ is symmetric with respect to the diagonal $AC$, if $AB = AD$, and $BC = CD$.

Ex. **4**. A regular hexagon is symmetric with respect to its center.

Ex. **5**. Two figures symmetric with respect to an axis are congruent.

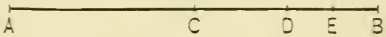
Ex. **6**. Two figures symmetric with respect to a point are congruent.

## THE INCOMMENSURABLE CASE BASED UPON LIMITS[1]

**457.** A **constant** is a quantity that maintains the same value throughout the same discussion. A **variable** is a quantity whose value changes during the same discussion.

Two quantities are **commensurable** if they have a common measure. Two quantities are **incommensurable** if they have no common measure.

**458.** If a variable $x$ approaches a constant $a$ so that the difference between $a$ and $x$ becomes and remains less than any conceivable number (not zero), then $a$ is called the **limit** of $x$.



For example, suppose a point $P$ to move from $A$ to $B$ in such a way as to move in the first second over half of $AB$ to $C$, in the second second, over half of the remainder, $CB$, to $D$, in the third second over half of the remainder, $DB$, to $E$, and so on indefinitely.

It is evident that the distance from $A$ to the moving point $P$ is a variable whose value can be made to differ from $AB$ by less than any assigned quantity, although it never can be made equal to $AB$.

$AB$ is therefore the **limit of the variable.**

Ex. (*a*) What is the limit of $.999 \cdots$ ?

(*b*) What is the limit of $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$ to infinity?

**459.** THEOREM. **If two variables, $x$ and $y$, are always equal and $x$ approaches $a$ as a limit, then $y$ approaches $a$ as a limit.**

**Proof.** $a - x = a - y$, and $a - x$ can be made less than any assignable number.

Hence, $a - y$ can be made smaller than any assignable number.

$$\therefore\ y \longrightarrow a.$$

**460.** COR. **If two variables are always equal and each approaches a limit, the limits are equal.**

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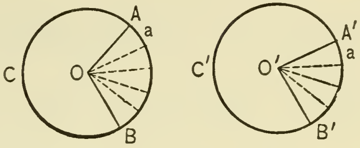
[1] This section is given for teachers who wish to introduce limits in connection with Book II. Otherwise it should be omitted. See note, p. 124.

## PROPOSITION I. THEOREM

**461.** *In the same circle, or in equal circles, two central angles have the same ratio as their arcs.*



**Given** in the equal circles $ABC$ and $A'B'C'$, two central angles $AOB$ and $A'O'B'$, having the arcs $AB$ and $A'B'$ respectively.

**To prove**
$$\frac{\angle AOB}{\angle A'O'B'} = \frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}.$$

**Proof.** CASE I. When the arcs are commensurable.

Let arc $a$ be a common measure contained in $\overset{\frown}{AB}$ $m$ times and in $\overset{\frown}{A'B'}$ $n$ times.

Then,
$$\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \frac{m}{n}.$$

Connect the points of division with the center. Then $\angle AOB$ will have been divided into $m$ parts and $\angle A'O'B'$ into $n$, all being equal. (183)

Whence
$$\frac{\angle AOB}{\angle A'O'B'} = \frac{m}{n}.$$

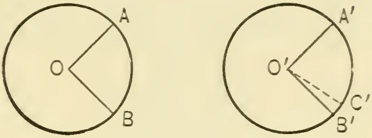
Hence if
$$\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \frac{m}{n},$$

then
$$\frac{\angle AOB}{\angle A'O'B'} = \frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}. \qquad \text{(Ax. 1.)}$$

NOTE. If the student cannot see the proof clearly by using the numbers $m$ and $n$, let him use some such ratios as 4 to 5, and then generalize his results.

CASE II. When the arcs are incommensurable.

Divide $\overset{\frown}{AB}$ into any number of equal parts, and apply one of those parts to $\overset{\frown}{A'B'}$ as many times as possible.



Since $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ are incommensurable, there must be a remainder $C'B'$ less than one of the equal parts.

Draw $O'C'$.

Since the arcs $AB$ and $A'C'$ are commensurable,

$$\frac{\overset{\frown}{A'C'}}{\overset{\frown}{AB}} = \frac{\angle A'O'C'}{\angle AOB}.$$

By increasing the number of parts into which $AB$ is divided, we can diminish the length of each part, and, therefore, the length of $C'B'$ indefinitely.

Hence $\overset{\frown}{A'C'}$ approaches $\overset{\frown}{A'B'}$ as a limit, and $\angle A'O'C'$ approaches $\angle A'O'B'$ as a limit.

Therefore

$$\frac{\overset{\frown}{A'C'}}{\overset{\frown}{AB}} \longrightarrow \frac{\overset{\frown}{A'B'}}{\overset{\frown}{AB}}, \text{ and } \frac{\angle A'O'C'}{\angle AOB} \longrightarrow \frac{\angle A'O'B'}{\angle AOB}.$$

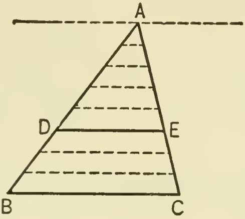
The variables $\dfrac{\overset{\frown}{A'C'}}{\overset{\frown}{AB}}$ and $\dfrac{\angle A'O'C'}{\angle AOB}$ being always equal, must have equal limits. (460)

Whence $$\frac{\overset{\frown}{A'B'}}{\overset{\frown}{AB}} = \frac{\angle A'O'B'}{\angle AOB}.$$

**462.** COR. *A central angle is measured by its intercepted arc.*

## PROPOSITION II. THEOREM

**463.** *A line parallel to one side of a triangle divides the other two sides into segments that are proportional.*



**Given** in $\triangle ABC$, $DE$ parallel to $BC$.

**To prove** $AD : DB = AE : EC$.

**Proof.** CASE I. $AD$ and $DB$ are commensurable.

Let $m$ be a common measure contained in $AD$ $a$ times and in $DB$ $b$ times.

Then $$\frac{AD}{DB} = \frac{a}{b}.$$

Through the points of division of $AB$ draw parallels to $BC$. These lines divide $AE$ into $a$ parts and $EC$ into $b$ parts, all being equal. (148)

Whence $$\frac{AE}{EC} = \frac{a}{b}.$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \quad \text{(Ax. 1.)}$$

CASE II. $AD$ and $DB$ are incommensurable.

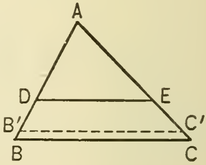
Divide $AD$ into any number of equal parts, and apply one of those parts to $DB$ as many times as possible.

As the lines $AD$ and $DB$ are incommensurable, there must be a remainder, $B'B$, less than one of the equal parts.

Draw $B'C'$ parallel to $BC$.

The lines $AD$ and $DB'$ are commensurable.

$$\therefore \frac{AD}{DB'} = \frac{AE}{EC'}.$$



By increasing the number of parts into which $AD$ is divided, we can diminish the length of these parts, and therefore the length of $B'B$ indefinitely.

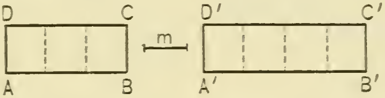
Hence $DB'$ approaches $DB$ as a limit, and $EC'$ approaches $EC$ as a limit.

The variables $\frac{AD}{DB'}$ and $\frac{AE}{EC'}$, being always equal, must have equal limits.

Whence $$\frac{AD}{DB} = \frac{AE}{EC}. \qquad (460)$$

## PROPOSITION III. THEOREM

**464.** *Rectangles having equal altitudes are to each other as their bases.*



Given in rectangles $ABCD$ and $A'B'C'D'$, altitude $AD$ = altitude $A'D'$.

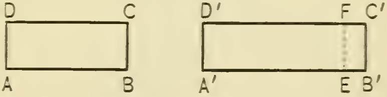
To prove $$\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}.$$

**Proof.** CASE I. $AB$ and $A'B'$ are commensurable.

Let $m$ be a common measure contained in $AB$ $a$ times and in $A'B'$ $b$ times.

[To be completed by the student. Compare (463).]

CASE II. $AB$ and $A'B'$ are incommensurable.



Divide $AB$ into any number of equal parts, and lay off one of these parts on $A'B'$ as many times as possible.

As $AB$ and $A'B'$ are incommensurable, there must be a remainder, $EB'$, less than one of the equal parts.

Draw $EF \perp A'B'$.

As $AB$ and $A'E$ are commensurable,

$$\frac{ABCD}{A'EFD'} = \frac{AB}{A'E}.$$

By increasing the number of parts into which $AB$ is divided, we can diminish the length of these parts, and, therefore, the length of $EB'$ indefinitely.

Hence, $A'E$ approaches $A'B'$ as a limit, and $A'EFD'$ approaches $A'B'C'D'$ as a limit.

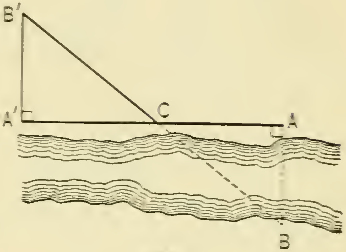
The variables $\frac{ABCD}{A'EFD'}$ and $\frac{AB}{A'E}$, being always equal, must have equal limits.

Whence
$$\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}.$$

# PRACTICAL APPLICATIONS OF PLANE GEOMETRY

## BOOK I[1]

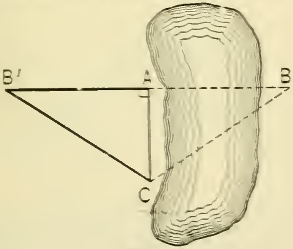
**1.** In order to determine the distance across a river $AB$, we measure $AA'$ at right angles to $AB$, and place a stake at $C$, the mid-point of $AA'$. From $A'$ we walk in a direction perpendicular to $AA'$ until we reach the point $B'$, which is in a straight line with $B$ and $C$. Which distance must we measure in order to obtain the width of the river? Why?



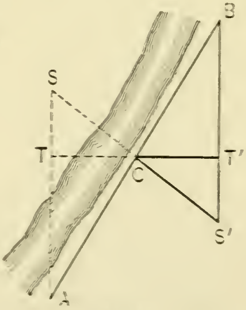
Ex. 1

**2.** Show what line we have to measure in order to determine the width of a lake $AB$, if we make $\angle ACB = \angle ACB'$ and $BB' \perp AC$.

**3.** In order to measure the distance of a tree from a steeple $S$, both on the other side of a river, I locate a point $A$



Ex. 2



Ex. 3

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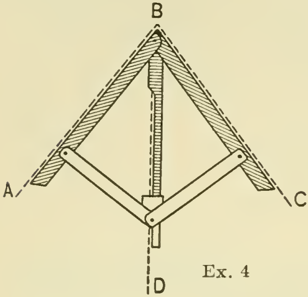
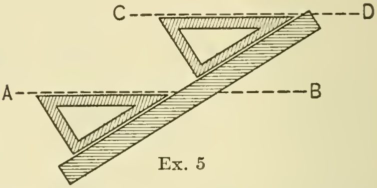
[1] It is strongly recommended that students measure some angles in the field by means of a transit, a plane table, a sextant, etc. It is not necessary that such instruments be very elaborate, and even very crude ones, made of two paper protractors, a ruler, and a few pins, are serviceable.

in the prolongation of $ST$, take a line $AB$, and place a stake at its mid-point $C$. From $B$, I walk in a direction $BS'$ so that $\angle B = \angle A$, and put stakes at point $T'$ which lies in the prolongation of $TC$, and at the point $S'$, which lies in the prolongation of $SC$. Which line must I measure? Prove your statement.



Ex. 4

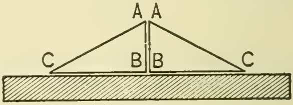


Ex. 5

4. Show how the annexed *Angle Divider* can be used to obtain the bisector $BD$ of any angle $ABC$.

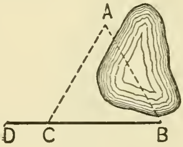
5. Show how a ruler and a triangle can be used to draw a line $CD$ parallel to a given line $AB$.

6. A triangle is placed against a ruler in the two positions which are indicated in the diagram, and the position of $AB$ is in both cases the same. How many degrees are in angle $B$?



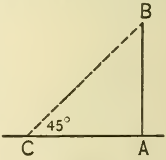
Ex. 6

7. To measure the distance $AB$, we walk from $B$ toward $D$ so that $\angle B = 60°$, until we reach $C$, a point at which $\angle ACB = 60°$. What line must we measure to obtain $AB$? Why?



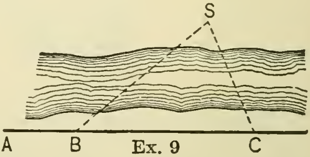
Ex. 7

8. At a point $C$, 80 ft. from the foot of a pole $AB$, the angle $ACB$ was found to be 45°. How high is the pole?



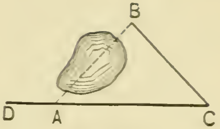
Ex. 8

9. Walking along the bank of a river from $A$ to $C$, a surveyor measures at $B$ the angle $ABS$, and walking 800 ft. further to $C$, he found angle $BCS$ to be one half of angle $ABS$. What is the distance $BS$?



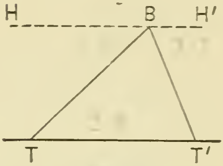
Ex. 9

**10**. Find the length $AB$ of a lake from the following measurements: $\angle DAB = 138°$, $\angle C = 42°$, $AC = 610$ yd., $BC = 400$ yd.



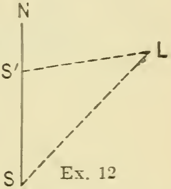
Ex. 10

**11**. A man in a balloon $B$ measures the angles which a horizontal line $HH'$ forms with the lines drawn to two towns $T$ and $T'$ as follows: $\angle HBT = 40°$, $\angle H'BT' = 70°$. If the distance between the towns is two miles, and if the balloon is directly above the line $TT'$, what is the distance of the balloon from $T$?



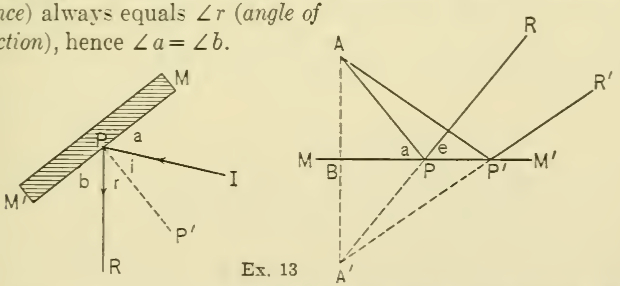
Ex. 11

**12**. A ship sailing north at the rate of 10 mi. per hour occupied at 8 A.M. the position $S$, and at 10 A.M. the position $S'$. How far was the ship at 10 A.M. from a lighthouse $L$, if $\angle S = 43°$ and $\angle NS'L = 86°$?



Ex. 12

**13**. In the diagram given here (and in the diagram of similar exercises) $MM'$ represents a mirror perpendicular to the plane of the paper, and all other lines represent lines *in* the plane of paper. If $P'P$ is $\perp M'M$, $IP$ is a ray of light and $PR$ is the reflected ray, then $\angle i$ (*angle of incidence*) always equals $\angle r$ (*angle of reflection*), hence $\angle a = \angle b$.

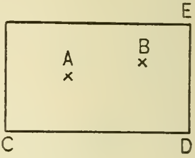


Ex. 13

If rays emitted from a point $A$ are reflected by a mirror $MM'$, prove that we obtain the reflected rays (*i.e.* make the necessary angles equal) by the following construction: Draw $AB \perp MM'$ and produce $AB$ by its own length to $A'$. Draw $A'P$. $PR$, the prolongation of $AP$, is the reflected ray.

**14.** A billiard ball is reflected by a cushion according to the same law as a ray of light is reflected by a mirror, *i.e.* the angle of incidence equals the angle of reflection. (Ex. 13.)

The billiard ball $A$ is to hit a point $P$ in the cushion $CD$, and be reflected therefrom so as to strike $B$. Determine the location of $P$ by a construction.



Ex. 14

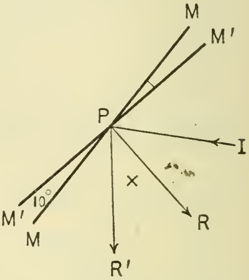
**15.** If the ball $A$ shall first strike cushion $CD$, then cushion $DE$, and finally hit $B$, construct the path of the billiard ball.

**16.** The ray $IP$ is reflected by $MM$ in the direction $PR$, and if the mirror is turned 10° in the position $M'M'$ ($\angle MPM' = 10°$), the same ray is reflected in the direction $PR'$. Find $\angle RPR'$.

**17.** Solve the same problem if

$$\angle MPM' = n°.$$

NOTE. The result of this problem forms the principle of the sextant, an instrument used chiefly by sailors, for measuring angles.

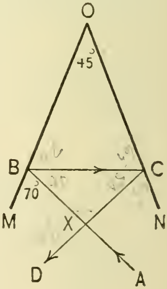


Ex. 16

**18.** Two mirrors $MO$ and $NO$ form an angle $O$ equal to 45°. If a ray $AB$ is reflected in the direction $BC$, and $BC$ in the direction of $CD$, find the angle formed by the first and last ray (*i.e.* $\angle x$), if

(*a*) $\angle ABM = 70°$.
(*b*) $\angle ABM = n°$.

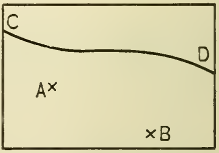
NOTE. The preceding exercise explains the principle of an instrument (called *optical square*) which is used to lay off right angles.



Ex. 18

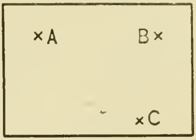
**19.** The annexed diagram represents a portion of a map, $A$ and $B$ two towns, and $CD$ a railroad.

It is proposed to build a station, $S$, which is just as far from $A$ as from $B$. Locate $S$ by a construction.



Ex. 19

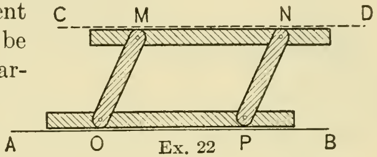
**20.** $A$, $B$, and $C$ represent the locations of three towns (on a map). It is proposed to build a common schoolhouse, $S$, which is equally far from $A$, $B$, and $C$. By a construction locate $S$.



Ex. 20

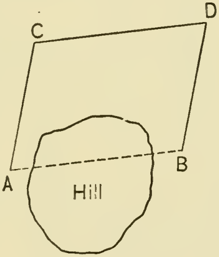
**21.** From a ship which sails N.E. (*i.e.* northeast) at the rate of 6 mi. an hour, a lighthouse is observed. At 9 P.M. the lighthouse appears exactly East, at 11 P.M. it appears exactly South. At what hour was the ship nearest to the lighthouse and what was the distance between them at this hour?

**22.** Explain how the instrument represented in the figure can be used to draw lines (*e.g.* $CD$) parallel to a given line $AB$ [$M$, $N$, $O$, and $P$ are pivots, $MO = NP$, and $MN = OP$].

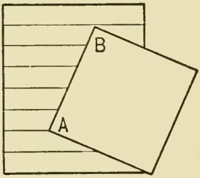


Ex. 22

**23.** To measure the distance from $A$ to $B$, two points on opposite sides of a hill, lay off two lines, $AC$ and $BD$, both running due north, and make $AC = BD$. Which line must be measured to obtain the length of $AB$? Why?



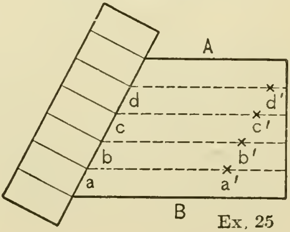
Ex. 23



Ex. 24

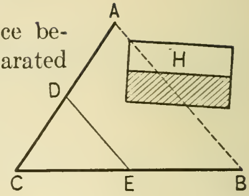
**24.** Explain how an ordinary ruled sheet can be used to divide a given line $AB$ in 5 equal parts. Under what conditions is it impossible to use this method?

**25.** To divide a strip $AB$ into 5 equal strips by lines parallel to $A$ and $B$, apply a ruled sheet as indicated in the figure and obtain the 4 points $a$, $b$, $c$, $d$. In like manner obtain $a'$, $b'$, $c'$, and $d'$, and join the corresponding points. Prove this construction.



Ex. 25

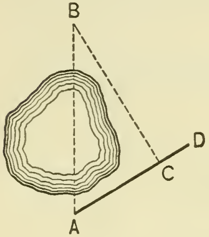
**26.** Show how to determine the distance between two points $A$ and $B$, which are separated by a house, by measuring the following lines: $AD = 9$ yd., $AC = 18$ yd., $EC = 11$ yd., $CB = 22$ yd., and $DE = 10$ yd.



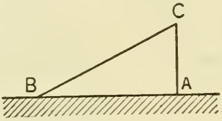
Ex. 26

**27.** To measure the distance $AB$, a man walks in the direction $AD$, so that $\angle BAD = 60^\circ$, and stops at a point $C$, where $\angle BCD = 90^\circ$. If $AC = 200$ yd., what is the length of $AB$?

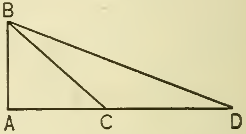
**28.** The top of a flagstaff, 75 ft. high, is partly broken and touches the ground at $B$. Find the height of $AC$, if $\angle B = 30^\circ$.



Ex. 27



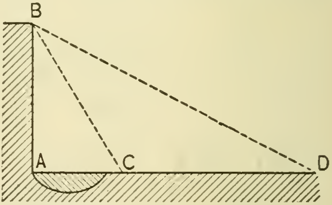
Ex. 28



Ex. 29

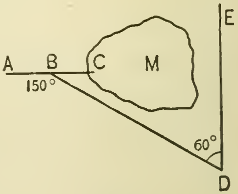
**29.** To measure the height, $AB$, of a tower, a man walks on a level straight road $DCA$ that leads directly to the foot of the tower. At $D$, $\angle BDA = 15^\circ$, and at $C$, $\angle BCA = 30^\circ$. Find $AB$, if $DC = 300$ ft.

**30.** $AB$ is a perpendicular cliff rising at one side of a river. An observer at a point $C$, 20 ft. from the opposite bank, measures $\angle ACB = 60^\circ$, and walking 200 ft. further away from the river to $D$, he measures $\angle BDA = 30^\circ$. Find the breadth of the river.



Ex. 30

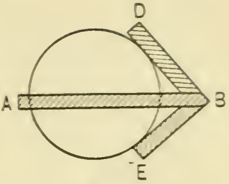
**31.** A straight railway $AB$ meets a mountain at $C$, and a tunnel is being driven at $C$ in the direction $AC$. It is proposed to commence this work at the other side of the mountain also. Angle $ABD$ is made equal to $150^\circ$, $BD = 3$ km., and $\angle D = 60^\circ$. How far from $D$ in $DE$ must the tunnel be driven, and what is its direction?



Ex. 31

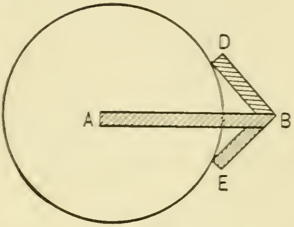
## BOOK II

**32**. The figure represents an instrument used for locating the center of circular disks. Three pieces of metal are so joined that the edge $AB$ bisects the angle formed by $BD$ and $BE$, $BD$ and $BE$ being equal. Prove that $AB$ passes through the center of the circle.



Ex. 32

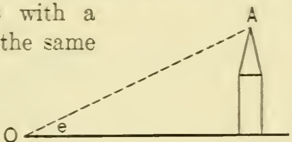
**33**. The next figure represents the same instrument applied to a much larger disk. Prove that $AB$ passes through the center.



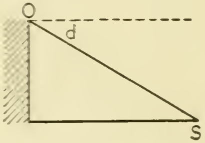
Ex. 33

**34**. The *angle of elevation* or *angle of depression* of any object $A$ as observed from $O$, is the angle which the line from the eye of the observer $O$ to the object makes with a horizontal line in the same vertical plane.

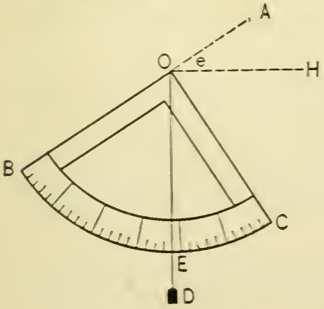
If the object lies higher than the observer, the angle is an angle of elevation, as $\angle e$. If the object lies lower than the observer, the angle is an angle of depression, as $\angle d$.



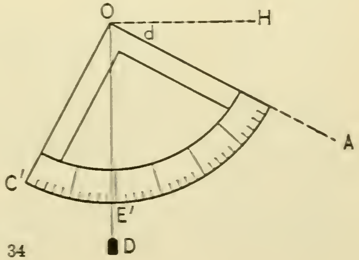
Ex. 34



Prove that the angle of elevation of an object may be measured as follows: A quadrant is placed in vertical position so that the pro-

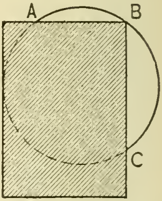


Ex. 34



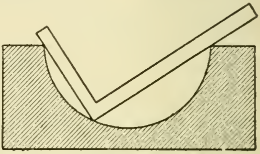
longation of $BO$ passes through $A$. (This is done by means of "sights" attached at $BO$.) A plumb line $OD$, held by a pin at $O$, intersects the arc $BC$ at $E$. The angle of elevation is measured by arc $EC$. Similarly arc $C'E'$ measures the angle of depression $d$.

**35.** Prove that a diameter of a circular disk may be drawn from $A$ as follows: Place a rectangular sheet of paper so that one vertex, $B$, lies on the circle, and the adjacent side passes through $A$. $C$ is the other end of the required diameter.

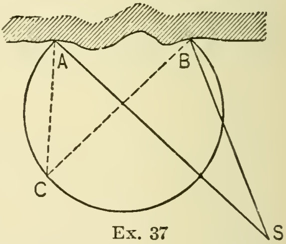


Ex. 35

**36.** Justify the following method for testing the accuracy of a semicircular groove: Place a carpenter's square in the position indicated in the figure. If its vertex touches every point of the groove as the square slides around, the groove is a true semicircle.


Ex. 36

**37.** If $A$ and $B$ are two lighthouses, and dangerous rocks lie within the circle $ABC$, while all points without the circle are navigable, prove that a ship $S$ is not in danger of striking the rocks as long as $\angle S$ is less than $\angle C$.



Ex. 37

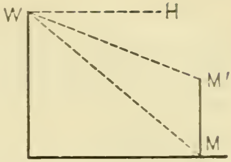
NOTE. The following six constructions (38 to 43) should be carried out by means of protractor and ruler. The required lines, or angles, should be found (approximately) by measurement. (See p. 303, Ex. 19.)

**38.** A vertical pole 25 ft. high casts a shadow 40 ft. long. What is the angle of elevation of the sun? (Let 1 in. represent 20 ft.)

**39.** A perpendicular cliff rises from the bank of a river. The angle of depression of the other bank, as observed from the top of the cliff, is 40°. If the cliff is 120 ft. high, what is the breadth of the river? (Let 1 in. represent 60 ft.)

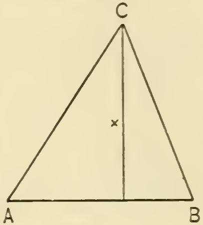
**40.** At a horizontal distance of 80 ft. from the foot of a tower, the angle of elevation of its top is 50°. Find the height of the tower.

**41.** From a window 60 ft. above the ground, the angles of depression of the top and the bottom of a monument are observed to be 20° and 40°. Find the height of the monument.



Ex. 41

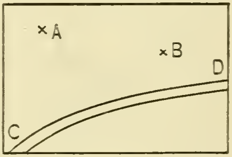
**42.** From two stations $A$ and $B$, on opposite sides of a cloud, the angles of elevation are found to be $A = 60°$, $B = 70°$. Find the height of the cloud if $AB = 2$ mi.



Ex. 42

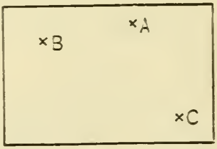
**43.** If the radius of the equator is 4000 mi., what is the radius of the parallel 40° N.? (Make 2000 mi. = 1 in.)

**44.** On a map $CD$ represents a road and $A$ and $B$ two steeples. A man walking on the road from $C$ to $D$ measured at a certain point $X$, the angle $AXB = 45°$. He also noticed that $X$ was nearer to $B$ than to $A$. Locate the point $X$ on the map.



Ex. 44

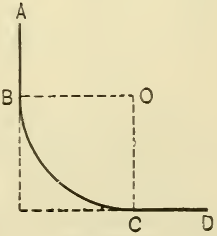
**45.** $A$, $B$, and $C$ are three points on a map. At a certain point $X$, a surveyor measured $\angle AXB = 90°$ and $\angle AXC = 60°$. Locate the point $X$ on the map.



Ex. 45

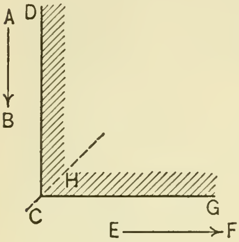
**46.** At the rectangular intersection of two streets the corner of the sidewalk is sometimes "rounded off." Construct the arc representing the rounded corner if its radius equals 25 ft. and if we assume the scale 1 : 500 (*i.e.* in the drawing 1 in. represents 500 in.).

**47.** Solve the preceding problem if the streets intersect at an acute angle.

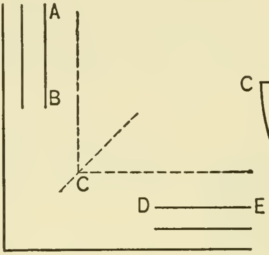


Exs. 46, 47

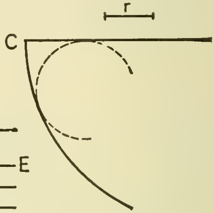
**48.** A wheelman riding in the direction *AB* (∥ *DH*) wishes to turn the corner by moving in a circular path so that the nearest approach to the sidewalk is indicated by the point *C*. He continues his ride in the direction *EF* (∥ *HG*). If *CH* bisects ∠ *H*, and the distance of the first parallel lines (*AB* and *DH*) is equal to the distance of the second pair (*HG* and *EF*), construct the arc representing his path.



Ex. 48



Ex. 49

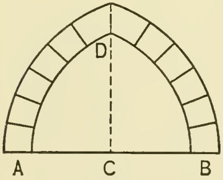


Ex. 50

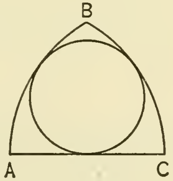
**49.** Solve the analogous problem for the track of a street car line.

**50.** "Round off" the corner *C*, made by a straight street and a circular one, by constructing a circle whose radius equals a given line *r*.

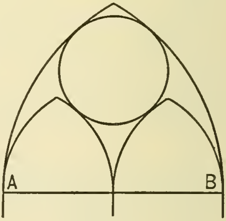
**51.** Construct a Gothic arch similar to the one in the diagram, if the span *AB* and the height *CD* are given. (The unknown centers of the arcs *AD* and *BD* lie in *AB*.)



Ex. 51



Ex. 52



Ex. 53

**52.** A Gothic arch *ABC* is equilateral, if *A* is the center of *BC* and *C* is the center of *AB*.

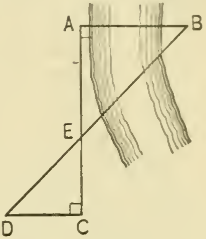
Inscribe a circle in an equilateral arch.

**53.** Construct the annexed figure, which consists of three equilateral arches and a circle touching the three, if *AB* is given.

## BOOK III

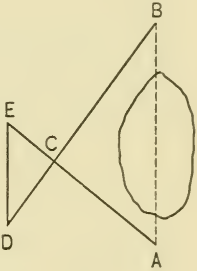
**54.** The shadow of a steeple upon level ground is 120 ft., while a pole 8 ft. long casts a shadow 3 ft. long. How high is the steeple?

**55.** To determine $AB$, the width of a river, measure $AC$ at right angles to $AB$, and $CD$ at right angles to $AC$. By sighting from $D$ to $B$ locate the point $E$. What is the length of $AB$ if $AE = 200$ ft., $EC = 25$ ft., $DC = 20$ ft.?



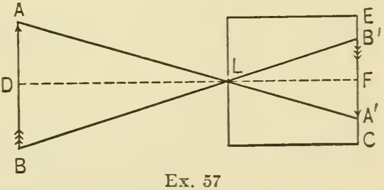
Ex. 55

**56.** Show how the distance $AB$, which cannot be measured directly, may be determined by the following measurements: $AE = 80$ ft., $EC = 20$ ft., $DB = 100$ ft., $DC = 25$ ft., $ED = 30$ ft.



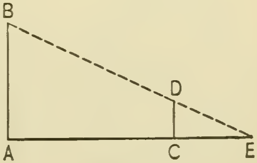
Ex. 56

**57.** The location of the image $A'$ of a point $A$, formed in a photographer's camera, is approximately found by drawing a straight line $AA'$ through the center of the lens $L$. If $CE$ is the position of the photographic plate, then $A'B'$ is the image of $AB$. How large is $A'B'$ if $AB = 6$ ft., $LD = 12$ ft., and $LF = 6$ in.?



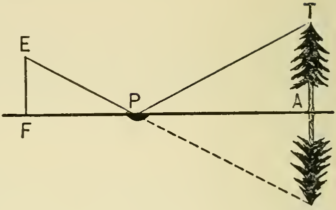
Ex. 57

**58.** The top of a tower $AB$ appears in a straight line with $D$, the top of a pole $DC$, when viewed from $E$. If $EA$ is horizontal, $EC = 4$ ft., $DC = 3$ ft., and $CA = 36$ ft., what is the height of the tower?



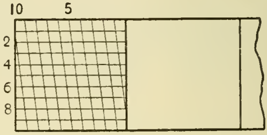
Ex. 58

**59.** An observer sees the reflected image of the top of a tree formed by a pool of water in the direction $ET'$. If $FPA$ is a horizontal straight line, both $EF$ and $TA$ are perpendicular to $FPA$, $EF = 5$ ft., $FP = 8$ ft., and $PA = 30$ ft., find the height of the tree.



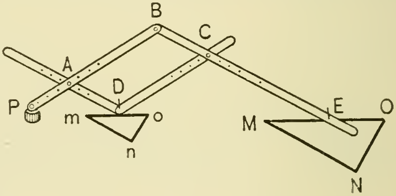
Ex. 59

**60.** If the side of the square in the diagram (*diagonal square*) is unity, show how we may measure .1, .2, .65, .34, etc.



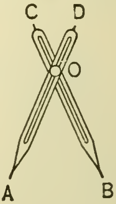
Ex. 60

**61.** A *pantograph* is a machine for drawing a figure $MNO$ similar to a given figure $mno$. It is used for enlarging and reducing maps and drawings. It consists of four bars, which are jointed at $A$, $B$, $C$, and $D$, and which are made parallel by making $AB = DC$ and $AD = BC$. $P$ turns about a fixed pivot, and pencils are carried at $D$ and $E$. The length of $PA$ and $CE$ is so adjusted that $\frac{PA}{AD} = \frac{DC}{CE}$. Prove that (1) $P$, $D$, and $E$ are in a straight line, (2) $\frac{PD}{PE} = \frac{PA}{PB}$, (3) any straight line $MN$ is parallel to $mn$ and $\frac{mn}{MN} = \frac{PA}{PB}$, (4) any triangle $MNO$ drawn by the pantograph is similar to the given triangle $mno$.



Ex. 61

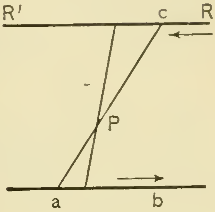
**62.** The figure represents an instrument (*proportional compasses*) for obtaining a line $CD$ which is any given fraction of a given line $AB$. By adjusting the length of $OC$, $OD$ we may make this fraction equal to $\frac{2}{5}$ of $AB$, etc. Prove that $CD = \frac{2}{5}$ of $AB$, if $OC = \frac{2}{5}\,OB$ and $OD = \frac{2}{5}\,OA$.



Ex. 62

**63.** A map is drawn to the scale of 1 in. to 100 mi. How far apart are two places that are $3\frac{1}{2}$ in. apart on the map?

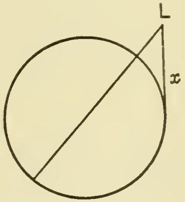
**64.** A man walking at the rate of 3 mi. per hour on a straight road *ab* which is parallel to a straight railroad *RR'* observes that a pole *P* appears in the same line of sight with a point *C* of a moving train. If the distance of *P* from *ab* is 10 ft. and the distance between *RR'* and *ab* is 100 ft., what is the rate of the moving train? Does it move at uniform speed?



Ex. 64

Note. The following examples which relate to the earth can often be simplified by bearing in mind that some numbers are only approximations. Thus, if a secant equals $8000\frac{26}{5280}$ mi., we may drop the fraction and make it equal to 8000, since the error in assuming the diameter of the earth as 8000 is much greater than the small fractions. Similarly we may measure the distance between two points on the earth's surface by a straight line instead of an arc, etc.

**65.** If the diameter of the earth is 8000 mi., how far can you see (*a*) from a lighthouse 96 ft. high? (*b*) From a mountain 2400 ft. high?



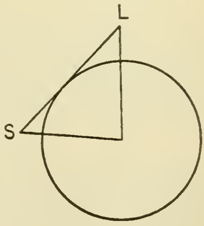
Ex. 65

**66.** Prove that *n*, the number of miles one can see at sea from an elevation of *h* ft., can be found approximately by the formula

$$n = \sqrt{\frac{3h}{2}}.$$

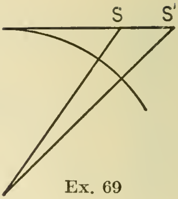
**67.** A swimmer places his eye at the surface of a smooth lake and finds that the top of a sailboat 4 mi. away is just visible. How high is the top of the sailboat above the water?

**68.** From the deck of a steamer 24 ft. above the sea, the top of a lighthouse that is 40 ft. high can just be seen beyond the horizon. Find the distance (*SL*) between steamer and lighthouse.



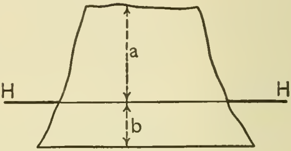
Ex. 68

**69.** From the deck of a steamer which is 54 ft. above sea level, the top of the mast of a sailboat appears in line with the horizon. If the sailboat is 2 mi. distant from the steamer, find the height of its mast.



Ex. 69

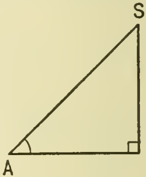
**70.** From the deck of a steamer which is 54 ft. above sea level, an iceberg is sighted and the height of its visible part is divided by the line of the horizon $HH'$ into two parts $a$ and $b$ so that $a = 3b$. If the distance of the iceberg from the steamer is 3 mi., find the height of the iceberg.



Ex. 70

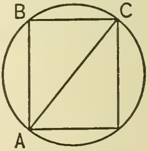
**71.** From a point 6 ft. above sea level the visible horizon has a radius of 3 mi. Find the diameter of the earth.

**72.** The Harder funicular railway near Interlaken (Switzerland) has a length of 1593 yd., and an inclination of 30°. If the lower station has an elevation of 1900 ft. above sea level, find the elevation of the upper station.



Ex. 73

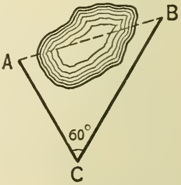
**73.** A skyrocket is seen to explode at an angle of elevation $A = 45°$. If the flash is seen $\frac{3}{4}$ sec. before the report of the explosion is heard, find the height of the skyrocket when it exploded. (Assume velocity of sound equal to 330 meters per second.)



Ex. 74

**74.** The strongest beam that can be cut from a given round log is the one in which the height $AB$ is to the width $BC$ as $\sqrt{2}:1$. Draw a diameter $AC$ and *construct* $B$ so that

$$AB : BC = \sqrt{2} : 1.$$

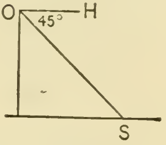


Ex. 75

**75.** What is the distance between two points $A$ and $B$ on opposite sides of a hill if $AC = 20$ rd., $BC = 32$ rd., and $\angle C = 60°$?

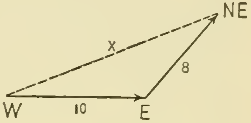
**76.** A vertical pole on a horizontal plane casts a shadow 50 ft. long when the angle of elevation (usually called *altitude*) of the sun equals 30°. Find the height of the pole.

**77.** From a cliff 300 ft. high, the angle of depression of a ship is 45°. Find the distance of the ship from the top of the cliff.



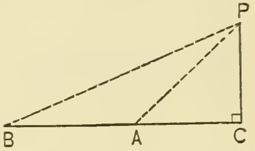
Ex. 77

**78.** A sailing vessel is propelled by the wind in an easterly direction at the rate of 10 mi. per hour. At the same time a current carries it N.E. at the rate of 8 mi. per hour. What is the true speed of the vessel per hour?



Ex. 78

**79.** The angle of elevation of an inaccessible point $P$, measured at $A$, equals 45°. At a point $B$, 500 ft. more distant, the angle of elevation of $P$ equals 22° 30′. If $BAC$ is a horizontal line, what is the elevation of $P$ above $A$ (*i.e.* $PC$)?

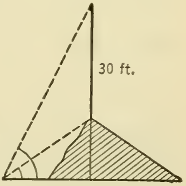


Ex. 79

**80.** From a balloon which is directly above one town the angle of depression of another town is found to be 30°. Find the height of the balloon if the towns are 3 mi. apart.

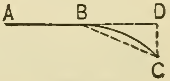
**81.** Two observers on opposite sides of a cloud measure the angles of elevation; $A = 45°$ and $B = 67° 30'$. Find the height of the cloud if $AB = 3$ mi.

**82.** A flagstaff 30 ft. high stands on top of a mound, and the angles of elevation of the top and the bottom of the pole are respectively 60° and 30°. Find the height of the mound.

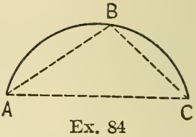


Ex. 82

**83.** The track $ABC$ consists of a straight line $AB$ and the arc $BC$. Find the radius of arc $BC$ if line $BC = 50$ yd., and the perpendicular $CD$ dropped upon the prolongation of $AB$ equals 4 yd.



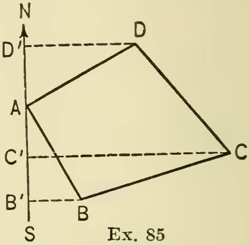
Ex. 83



Ex. 84

**84.** A circular track is to be constructed so as to pass through the points $A$, $B$, and $C$. If $AB = 50$ yd., $BC = 30$ yd., and $AC = 70$ yd., find the radius of the arc $ABC$.

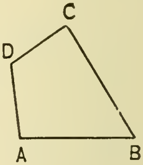
## BOOK IV



Ex. 85

**85.** To measure the area of a quadrangular field $ABCD$, a surveyor measures the perpendiculars drawn to a line $NS$ and the segments made on $NS$ as follows: $AD' = 6$ rd., $AC' = 5$ rd., $C'B' = 4$ rd., $D'D = 7$ rd., $C'C = 12$ rd., and $B'B = 4$ rd. Find the area of $ABCD$.

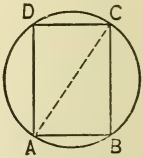
**86.** Determine the area of a pentagonal field $ABCDE$, if $AB = 18$, $BC = 24$, $CD = 28$, $DE = 24$, $EA = 10$, $AC = 30$, and $AD = 26$.



Ex. 87

**87.** An irregular quadrangular field $ABCD$ must be divided into three equivalent parts by lines passing through $A$. Construct the lines of division.

**88.** On a map whose scale is 1 : 500, how many square feet are represented by a polygon equivalent to 3 sq. in.?



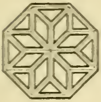
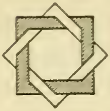
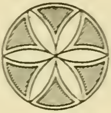
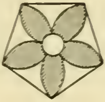
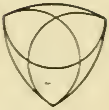
Ex. 89

**89.** From a given round log a rectangular beam is to be cut so as to make its cross section $ABCD$ as large as possible. Construct $ABCD$.

**90.** A triangular field must be divided into 4 equivalent parts by lines parallel to one side. Construct the lines of division.

## BOOK V

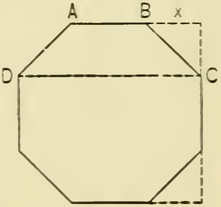
**91.** Construct the following patterns.

**92.** It is claimed that a rectangle (*e.g.* an envelope, a window, a picture, etc.) is most pleasing to the eye if its length and its width have the same ratio as the segments of a line which is divided in extreme and mean ratio. If a window is 4 ft. wide, what should be its height, if the two dimensions have the above-mentioned ratio?

**93.** The side of a tower whose base is a regular hexagon measures 10 ft. Find the area of the ground occupied by the tower.

**94.** $AB$, the side of the base of a regular octagonal tower, equals 8 ft. Find $CD$, the distance between two opposite walls.



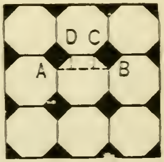
Ex. 94

**95.** Find the area of the ground covered by the tower described in the preceding exercise.

**96.** The linoleum pattern in the figure consists of regular octagons (white) and squares (black). If the side of the squares equals 2 in., what is the length of $AB$, the distance between two opposite sides of an octagon?

**97.** In the same figure find the area of an octagon ($AD = 2$ in.).

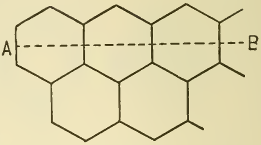
**98.** If the length of $AB$ is given, construct the pattern of the preceding exercise.



Ex. 96

**99.** If the length of $AB$ (preceding figure) equals 6 in., calculate the side of a square.

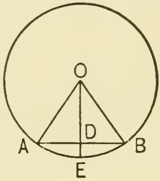
**100.** The floor of a room whose length is $AB$ is covered with regular hexagonal tiles. How many tiles are in one row that extends over the entire length ($AB$) of the room, if each side of the hexagons equal 2 in., $AB = 10$ ft., and the tiles are placed in the position indicated in the figure?



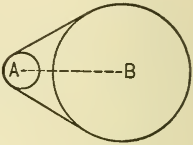
Ex. 100

**101.** The approximate value of the circumference of a circle is sometimes found by carpenters as follows: Draw the equilateral triangle $AOB$ and produce its altitude $OD$ to $E$. The circumference equals $6\,AO + 2\,DE$. Find the error if $\pi$ is equal to 3.1416.

**102.** Two wheels $A$ and $B$ are connected by a belt, and their radii are respectively 2 ft. and 9 ft. If the larger wheel ($B$) makes 40 revolutions per minute, how many revolutions per minute will the smaller wheel make?



Ex. 101

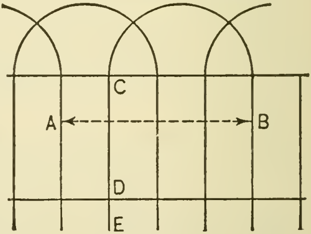


Ex. 102

**103.** In the figure of the preceding exercise find the length of the belt if $AB = 14$ ft. (Radii are 2 ft. and 9 ft.)

**104.** The gauge of an automobile (*i.e.* the distance between two wheels on the same axis) equals 4 ft. $8\frac{1}{2}$ in. How many inches more than an inner wheel does an outer wheel travel when the car turns around a rectangular corner?

**105.** The circumference of the equator is (approximately) 25,000 mi. Imagine a concentric circle whose circumference is 1 ft. longer. What is the difference of the radii of the two circles?

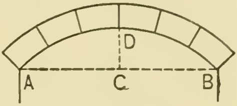


Ex. 106

**106.** The figure above represents a small wire fence used to protect flower beds, etc. How many feet of wire are needed per running foot of fence if $AB = 1$ ft., $CD = 9$ in., $DE = 3$ in.?

**107.** If two streets meet at an angle of 120°, and an automobile turns from one into the other, what is the difference between the distances traveled by an outer and an inner wheel of an automobile? (Gauge 4 ft. $8\frac{1}{2}$ in.)

**108.** The span of a circular arch $AB$ equals 10 ft., and its height $CD = 2$ ft. Find the radius of the arch.



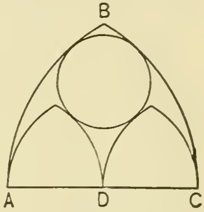
Ex. 108

**109.** If the earth's orbit be assumed to be a circle whose radius = 93,000,000 mi., and the year = 365 da., how many miles per hour does the earth travel?

**110.** Find the area of an equilateral Gothic arch $ABC$, if $AC = 4$ ft.

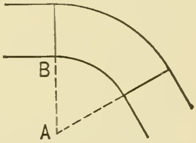
**111.** In the same figure, what is the area of each of the smaller equilateral arches if $AD = DC = 2$ ft.?

**112.** In the same diagram, what is the area of the circle that touches the three equilateral arches?



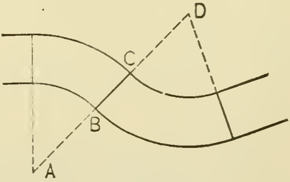
Exs. 110–112

**113.** A sidewalk 5 ft. wide turns a corner. Find the area of the curved portion of the sidewalk if the radius $AB = 8$ ft. and $\angle A = 60°$.



Ex. 113

**114.** Determine the area of the curved portion of the road drawn in the figure, if $AB = 90$ ft., $BC = 50$ ft., $CD = 90$ ft., $\angle A = 45°$, and $\angle D = 60°$.

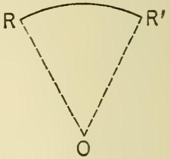


Ex. 114

**115.** How many miles per hour does a point whose latitude equals 45° travel in consequence of the earth's rotation about its axis $P$ if we assume the diameter of the earth equal to 8000 mi., and the time of one rotation equal to 23 hr. 56 min.?

**116.** It is found that a water pipe whose diameter is 2 in. supplies half the amount of water that is needed in a building. What would be the diameter of a pipe that would supply the entire required amount of water?

**117.** What should be the diameter of a water pipe that supplies the same amount of water as both an 8-in. and a 6-in. pipe, if we assume that all other conditions, as pressure, etc., are the same in the three pipes?

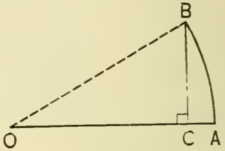


Ex. 118

**118.** A rail 50 ft. long must be bent through what angle (*i.e.* ∠ $O$ formed by perpendiculars at its end) if the radius of curvature equals 360 ft.?

**119.** In laying a curved track, a rail 55 ft. long is bent through an angle of $17° 30'$. Find the radius of curvature.

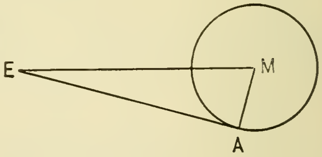
**120.** If $O$ is the center of arc $AB$, $BC \perp OA$, and $\angle O$ is small, then the difference between $\widehat{AB}$ and $BC$ is very small. If we have to find the value of $BC$, we may, in such cases, find the length of arc $AB$ instead. If $\angle O = 4°$, the error due to this simplification equals .00005 of the result, and for smaller angles the error is much smaller, since it is approximately proportional to the cube of the angle.



Ex. 120

If $BC$ is a tower standing on a horizontal plane $AO$, $BO = 2000$ ft. and $\angle O = 2°$. Find the height of the tower.

**121.** The apparent diameter of the moon equals $30'$, *i.e.* a line $EM$ drawn from earth ($E$) to the center of the moon ($M$) and a tangent $EA$ include an angle of $15'$. If $EM = 243{,}000$ mi., find the radius of the moon.



Ex. 121

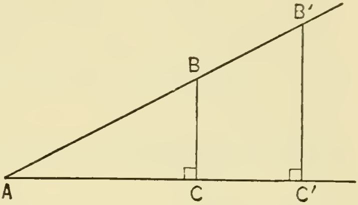
**122.** The apparent diameter of the sun is approximately the same as that of the moon (preceding problem). What is the diameter of the sun if its distance from the earth equals 93,000,000 mi.?

**123.** Applying Trigonometric Functions (pp. 301–304), solve problems 75–82.

**124.** By the same method, solve problems designated in 123, first replacing ∠ 30° by ∠ 33°, ∠ 60° by ∠ 64°, and ∠ 45° by ∠ 40°.

## TRIGONOMETRIC FUNCTIONS

**465.** In order to be able to solve many problems involving angles different from 60° and 90° and their multiples and submultiples, we have to turn to the trigonometric functions of angles. A trigonometric function of an angle is a ratio that depends upon the angle for its value.



Let $\angle A$ be given and $BC$ a fixed line drawn from a point in one side of the angle perpendicular to the other side ($BC \perp AC$). From any other point as $B'$ draw the line $B'C' \perp AC$.

Then right $\triangle ABC \sim$ right $\triangle AB'C'$. (Why?)

$$BC : B'C' = AB : AB' \qquad (303)$$

and
$$BC : AB = B'C' : AB' \qquad (279)$$

or
$$\frac{BC}{AB} = \frac{B'C'}{AB'}.$$

But $\dfrac{BC}{AB}$ is constant for a fixed $\angle A$. (Why?)

Hence $\dfrac{B'C'}{A'B'}$ = the same constant for every position of $B'$. (Why?)

**466.** This ratio is called the sine of angle $A$.

Likewise the ratio $\dfrac{AC}{AB}$ is called the cosine of angle $B$.

Also the ratio $\dfrac{BC}{AC}$ is called the tangent of angle $A$.

The numerical values of these ratios have been found by more advanced mathematics and placed in tables convenient for our use. Acute angles only are considered in this textbook.

**467.** DEF. The **sine** of an angle equals $\frac{\text{side opp. the angle}}{\text{hypotenuse}}$.

The **cosine** of an angle equals $\frac{\text{side adjacent to the angle}}{\text{hypotenuse}}$.

The **tangent** of an angle equals $\frac{\text{side opp. the angle}}{\text{side adj. to the angle}}$.

Using the table (p. 304), work out the following exercises:

Ex. **1.** What angle has its sine equal to .326? To .500? To .946? To .993? To .755?

Ex. **2.** What angle has its cosine equal to .970? To .707? To .309? To .883? To. 438?

Ex. **3.** What angle has its tangent equal to .158? To 2.145? To 5.671? To .625? To .249?

Ex. **4.** Find, to the nearest degree, the angle having its sine equal to .450.

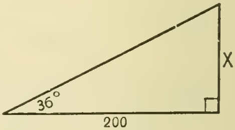
Ex. **5.** Find, to the nearest degree, the angle having its cosine equal to .450.

Ex. **6.** Find, to the nearest degree, the angle having its tangent equal to .290.

Ex. **7.** From the table, as the angle increases, how do its three functions behave?

Ex. **8.** The angle of elevation (see p. 287, Ex. 34) of the top of a tower 200 feet from its base is 36°. Find the height of the tower.

HINT. $\frac{x}{200} = \tan 36°$.



Ex. 8

Ex. **9.** The angle of depression (see p. 287, Ex. 34) of a boat from the top of a lighthouse 84 feet high is 17°. How far is the boat from the foot of the lighthouse? HINT. $\tan 17° = \frac{84}{d(\text{istance})}$.

Ex. **10**. A balloon is held to the ground by a taut wire rope that is 328 feet long. If the rope makes an angle of 67° with the ground, how high is the balloon?

Ex. **11**. In the following problems, the right triangle has its right angle at *C*. *a* and *b* are the sides opposite angles *A* and *B* respectively, and *c* is the hypotenuse. Substitute for the question marks.

| $\angle A$ | $\angle B$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 48° | ? | ? | ? | 280 |
| ? | 37° | 14.2 | ? | ? |
| ? | ? | 8.3 | 10.2 | ? |
| ? | ? | ? | 6.5 | 12.4 |
| 71° | ? | $3\frac{1}{3}$ | ? | ? |

Ex. **12**. Find the area of a right triangle given the hypotenuse 18 and one acute ∠40°.

Ex. **13**. How many square feet are there in the area of a triangle having two sides and their included angle respectively 41 feet, 36 feet, and ∠38°?

Ex. **14**. From a point *C* on one bank of a river, exactly opposite a tree *B* on the other bank, I walk at right angles to *CB* to a point *A* so that *CA* equals 120 yards. I find the angle *CAB* is 28°. What is the width of the river?

Ex. **15**. From a point *P* two tangents are drawn to a circle. These tangents are each 45 feet long and form an angle of 68°. Find the diameter of the circle.

Ex. **16**. The base angles of an isosceles triangle are each 34°. If the base is 80 ft., what is the altitude of the triangle?

Ex. **17**. The angle formed by the two radii drawn to the extremities of a chord is 130°. If the radii are 11 feet, how long is the chord?

Ex. **18**. The altitude on the hypotenuse of a right triangle is 14 feet. One angle of the triangle is 24°. Find the length of the hypotenuse.

Ex. **19**. Solve problems 38–42 in Practical Applications (pages 288–289).

Ex. **20**. Solve problems 123, 124, on page 300.

## VALUES OF TRIGONOMETRIC FUNCTIONS

| Degree | Sine ∠ | Cos ∠ | Tan ∠ | Degree | Sine ∠ | Cos ∠ | Tan ∠ |
|---|---|---|---|---|---|---|---|
| **0** | .000 | 1.000 | .000 | **45** | .707 | .707 | 1.000 |
| **1** | .017 | .999 | .017 | **46** | .719 | .695 | 1.036 |
| **2** | .035 | .999 | .035 | **47** | .731 | .682 | 1.072 |
| **3** | .052 | .998 | .052 | **48** | .743 | .669 | 1.111 |
| **4** | .070 | .997 | .070 | **49** | .755 | .656 | 1.150 |
| **5** | .087 | .996 | .087 | **50** | .766 | .643 | 1.192 |
| **6** | .105 | .995 | .105 | **51** | .777 | .629 | 1.235 |
| **7** | .122 | .993 | .123 | **52** | .788 | .616 | 1.280 |
| **8** | .139 | .990 | .140 | **53** | .799 | .602 | 1.327 |
| **9** | .156 | .988 | .158 | **54** | .809 | .588 | 1.376 |
| **10** | .174 | .985 | .176 | **55** | .819 | .574 | 1.428 |
| **11** | .191 | .982 | .194 | **56** | .829 | .559 | 1.483 |
| **12** | .208 | .978 | .213 | **57** | .839 | .545 | 1.540 |
| **13** | .225 | .974 | .231 | **58** | .848 | .530 | 1.600 |
| **14** | .242 | .970 | .249 | **59** | .857 | .515 | 1.664 |
| **15** | .259 | .966 | .270 | **60** | .866 | .500 | 1.732 |
| **16** | .276 | .961 | .287 | **61** | .875 | .485 | 1.804 |
| **17** | .292 | .956 | .306 | **62** | .883 | .469 | 1.881 |
| **18** | .309 | .951 | .325 | **63** | .891 | .454 | 1.963 |
| **19** | .326 | .946 | .344 | **64** | .899 | .438 | 2.050 |
| **20** | .342 | .940 | .364 | **65** | .906 | .423 | 2.145 |
| **21** | .358 | .934 | .384 | **66** | .914 | .407 | 2.246 |
| **22** | .375 | .927 | .404 | **67** | .921 | .391 | 2.356 |
| **23** | .391 | .921 | .424 | **68** | .927 | .375 | 2.475 |
| **24** | .407 | .914 | .445 | **69** | .934 | .358 | 2.605 |
| **25** | .423 | .906 | .466 | **70** | .940 | .342 | 2.748 |
| **26** | .438 | .899 | .488 | **71** | .946 | .326 | 2.904 |
| **27** | .454 | .891 | .510 | **72** | .951 | .309 | 3.078 |
| **28** | .469 | .883 | .532 | **73** | .956 | .292 | 3.271 |
| **29** | .485 | .875 | .554 | **74** | .961 | .276 | 3.487 |
| **30** | .500 | .866 | .577 | **75** | .966 | .259 | 3.732 |
| **31** | .515 | .857 | .601 | **76** | .970 | .242 | 4.011 |
| **32** | .530 | .848 | .625 | **77** | .974 | .225 | 4.332 |
| **33** | .545 | .839 | .649 | **78** | .978 | .208 | 4.705 |
| **34** | .559 | .829 | .675 | **79** | .982 | .191 | 5.145 |
| **35** | .574 | .819 | .700 | **80** | .985 | .174 | 5.671 |
| **36** | .588 | .809 | .727 | **81** | .988 | .156 | 6.314 |
| **37** | .602 | .799 | .754 | **82** | .990 | .139 | 7.115 |
| **38** | .616 | .788 | .781 | **83** | .993 | .122 | 8.144 |
| **39** | .629 | .777 | .810 | **84** | .995 | .105 | 9.514 |
| **40** | .643 | .766 | .839 | **85** | .996 | .087 | 11.430 |
| **41** | .656 | .755 | .869 | **86** | .997 | .070 | 14.300 |
| **42** | .669 | .743 | .900 | **87** | .998 | .052 | 19.081 |
| **43** | .682 | .731 | .933 | **88** | .999 | .035 | 28.636 |
| **44** | .695 | .719 | .966 | **89** | .999 | .017 | 57.290 |

## A SKETCH OF THE HISTORY OF GEOMETRY

**468.** Of course, the history of geometry cannot be traced actually to its beginning. No records were made at first and, no doubt, most of the earliest records are entirely lost to the world. Painstaking investigations make probable most of the statements that follow here.

**469.** The ancient **Babylonians** and **Chaldeans**, on account of their religious observances, made some study of geometry. Most of their knowledge was got merely through experiences in a practical way.

**470.** It is, however, to the ancient **Egyptians** that we owe the beginnings of geometry that spread through the countries of Europe, Western Asia, and the Americas. The necessity of building pyramids and temples, of the re-division of land after an overflow of the Nile, caused their engineers to collect empirical rules, which they applied in their various labors.

The oldest known manuscript dates back possibly to 1700 B.C. It was written by **Ahmes**, who says it was based upon an earlier work. Among other mathematical ideas, it contains rules on mensuration.

These Egyptians knew how to find the area of a rectangle exactly, but they thought the area of an isosceles triangle could be found by taking one half its base times one of its legs. They used 3.16 as a value for $\pi$. They knew how to lay off a right angle by using a right triangle having its three sides in the ratio of $3:4:5$. Of the logical foundations of geometry, they knew very little indeed.

**471.** The ancient **Chinese** and **Japanese** probably assumed $\pi$ to equal either 3 or $\sqrt{10}$. They had a few simple rules for mensuration based upon observation.

Likewise the ancient **Hindus** knew but little of geometry, most of which they probably got in the same manner as other ancient races. In later times the Hindus imported their ideas from the Greeks and, although they used the elementary principles of geometry, they gave no real logical proofs.

**472.** For the beginnings of demonstrative geometry we are indebted to the spirit of Greek investigation. The **Greeks** studied in Egypt, became acquainted with their simple rules in their applications, but went further and gave to the world the logical principles underlying elementary geometry.

To **Thales** (c. 575 B.C.), although he left no written records, is attributed the introduction of geometry into Greece. He knew that (1) vertical angles are equal, (2) the base angles of an isosceles triangle are equal, (3) two triangles are congruent if they have a side in each equal and the angles at the extremities of these sides respectively equal, (4) the diameters of a circle bisect the circle, (5) and, possibly, triangles that are mutually equiangular are similar.

He was the first Greek known to apply geometry to practical problems.

To **Pythagoras** (c. 540 B.C.) is ascribed the Pythagorean theorem: The square on the hypotenuse equals the sum of the squares on the other two sides. However, the earliest proof of this theorem that has come down to us is to be found in **Euclid**. The School of Pythagoras made geometry the foundation of their teachings. Of the various theorems that they proved may be mentioned: (1) the sum of the angles of a triangle equals two right angles, (2) to construct a parallelogram equivalent to a triangle, and (3) the diagonal of a square is incommensurable with a side. They did not add anything to the knowledge of the circle.

This school proved the various theorems by the *deductive* method.

The **Sophists** studied the circle and attempted to solve the so-called three famous problems of antiquity: (1) the trisection of any angle, (2) the duplication of the cube, and (3) the squaring of a circle. Although they failed completely in using elementary methods, they were led to discover many properties of circles.

**Hippocrates** (c. 460 B.C.) contributed much to the geometry of circles, applied logic in his proofs, and added theorems on similar figures and proportion. He wrote a textbook called the *Elements*.

**Plato** (c. 380 B.C.) and his school trained men's minds for correct and vigorous thinking, demanded that the straight edge and compasses be the only instruments for constructions, and introduced *analysis* as a method to be used in geometrical proofs.

**Aristotle** (c. 340 B.C.), although not much interested in mathematics as such, gave principles for forming logical definitions.

**Euclid** (c. 300 B.C.) is the author of the greatest textbook in the world. Other textbooks had been written on geometry, but it remained for Euclid to systematize the work of his predecessors, to remove imperfections, and to put on a firm logical basis the propositions in elementary geometry.

**Archimedes** (c. 225 B.C.) was the greatest mathematical physicist of antiquity. In his works we find the beginnings of much of the modern mathematics. He found that $\pi$ must lie within the limits of $3\frac{1}{7}$ and $3\frac{10}{71}$.

**473**. The **Romans** carried on almost no independent study of theoretical geometry. What they obtained from the Greeks they applied to their practical problems in engineering. They even used $3\frac{1}{8}$ as a value for $\pi$, since it fitted in with their duodecimal system of notation of numbers.

**474**. The **Arabs** added little to the geometry known in their day. However, by translating many textbooks into their language and thereafter transmitting into Europe the

Greek geometry, they did a signal service for mankind. In a later period the Arabs reduced many geometric problems to the solution of algebraic equations. This helped to unify mathematics.

**475.** During the **Middle Ages** Euclid held sway, and it was not until **Legendre**, a Frenchman, in 1794 wrote his *Elements of Geometry*, that the world began to use textbooks in geometry simply enough written to be understood by the students of lower schools. To-day we find many new kinds of geometry; such as non-Euclidean, projective, descriptive, hyper-space, and analytic.

**476.** It is hoped that many students may become sufficiently interested through this sketch to desire to read more about the history of mathematics. Excellent material will be found in Cajori's *A History of Mathematics.*

## NUMERICAL COMPUTATION

**477.** Approximation to nearest hundredth.

Ex. Find to nearest hundredth the values of $\frac{1}{8}$, $1\frac{2}{3}$, .63755, $\frac{9.217}{4}$.

$\frac{1}{8}$ = .125 and, to the nearest hundredth, is .13.
$1\frac{2}{3}$ = 1.666 and, to the nearest hundredth, is 1.67.
.63755 to the nearest hundredth is .64.
$\frac{9.217}{4}$ = 2.304 and, to the nearest hundredth, is 2.30.

**478.** The square roots of numbers.

Ex.

$$\begin{array}{r|l}
 & \;4.97- \\
 & \sqrt{24.7560} \\
 & \;16 \\
89 & \;8\,75 \\
 & \;8\,01 \\
987 & \;6460 \\
 & \;6909
\end{array}$$

*Ans.* 4.97 to nearest hundredth.

Simplification of radicals.

Ex. Rationalize $\frac{2}{\sqrt{3}}$ and find approximate value to the nearest hundredth.

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2(1.732)}{3} = \frac{3.464}{3} = 1.154^{+}.$$

Its value to nearest hundredth is 1.15.

**479**. The equation in geometry.

Ex. **1**. Find the angle whose supplement is 32° more than two and one half times its complement.

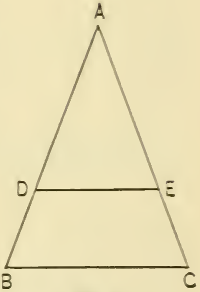
Sol.
$$\begin{aligned}180 - x &= 2\tfrac{1}{2}(90 - x) + 32.\\ 180 - x &= \tfrac{5}{2}(90 - x) + 32.\\ 360 - 2x &= 450 - 5x + 64.\\ 3x &= 154.\\ x &= 51\tfrac{1}{3}^\circ \text{ or } 51^\circ\ 20'.\end{aligned}$$

Ex. **2**. Given $AB = 10$, $AE = 8$, $EC = 5$, and $DE \parallel BC$, to find $AD$ and $DB$ in $\triangle ABC$.

Sol. Let $x = AD$, then $10 - x = DB$,

but $AD : DB = AE : EC$

or $x : 10 - x = 8 : 5.$
$$\begin{aligned}5x &= 8(10 - x)\\ 5x &= 80 - 8x.\\ 13x &= 80.\\ x &= 6\tfrac{2}{13} = AD.\\ 10 - x &= 3\tfrac{11}{13} = DB.\end{aligned}$$



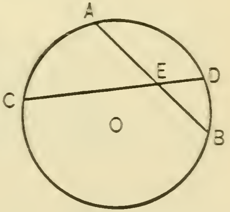
Ex. 2

Ex. **3**. In circle $O$, $CD$ is cut by $AB$ so that $CE = 12$ and $ED = 2$ If $AB = 11$, find segments $AE$ and $EB$.

Sol. $CE \cdot ED = AE \cdot EB.$

Let $x = AE$, then $11 - x = EB.$

Then
$$\begin{aligned}12 \times 2 &= x(11 - x).\\ 24 &= 11x - x^2.\\ x^2 - 11x &= -24.\\ x^2 - 11x + (\tfrac{11}{2})^2 &= \tfrac{121}{4} - 24 = \tfrac{121}{4} - \tfrac{96}{4} = \tfrac{25}{4}.\\ (x - \tfrac{11}{2})^2 &= \tfrac{25}{4}.\\ x - \tfrac{11}{2} &= \sqrt{\tfrac{25}{4}} = \pm\tfrac{5}{2}.\\ x &= \pm\tfrac{5}{2} + \tfrac{11}{2} = 8 \text{ or } 3.\\ 11 - x &= 3 \text{ or } 8.\end{aligned}$$
$$\therefore AE = 8 \text{ and } EB = 3, \text{ or } AE = 3 \text{ and } EB = 8.$$



Ex. 3

Ex. **4**. Find the radius of a circle having its circumference equal to the sum of three circumferences of circles with radii 3, 4, and 5 respectively.

Sol. Let $r$ = radius of required circle.

Then $2\pi r = 2\pi(3) + 2\pi(4) + 2\pi(5)$,

and $r = 3 + 4 + 5 = 12$.

Ex. **5**. Given a circle of radius 4 in., to find a side of an equivalent equilateral triangle.

Sol. Let $a$ = side of required triangle.

Then
$$\frac{a^2}{4}\sqrt{3} = \pi r^2 = 16\pi.$$
$$a^2 = \frac{64\pi}{\sqrt{3}}.$$
$$a^2 = \frac{64\pi\sqrt{3}}{3} = \frac{64(3.1416)(1.732)}{3}.$$
$$a^2 = 116.08$$
$$a = \sqrt{116.08} = 10.8^- \text{ in.}$$

## TABLE OF THE MOST IMPORTANT FORMULAS

### NOTATION

$a, b, c$ = sides of $\triangle ABC$.
$m_a$ = median to $a$.
$h_a$ = altitude to $a$.
$s = \frac{1}{2}(a + b + c)$.
$b$ = base.
$c$ = circumference.
$s_n$ = side of regular polygon of $n$ sides.
$b'$ = projection of $b$ upon $c$.
$R$ = radius of circumscribed ⊙.
$t_c$ = bisector of angle opp. **c**.
$r$ = radius of inscribed ⊙, or apothem.
$S$ = area (surface units).
$d$ = diameter; also diagonal of a square or rhombus.
$b_1$ and $b_2$ = bases of a trapezoid.
$p$ = perimeter.

## FORMULAS FOR LINE VALUES

Right △, $a^2 + b^2 = c^2$ ($c$ = hy.). (329)

Oblique △, $a^2 = b^2 + c^2 \pm 2\,b'c$, dependent upon $\angle A$ being obtuse or acute. (334, 336)

Circle, $c = 2\,\pi R = \pi\,d$. (424)

Diagonal of a square, $d = b\sqrt{2}$.

Equilateral △, $h = \frac{b}{2}\sqrt{3}$. (p. 189, Ex. 4.)

$s_3 = R\sqrt{3}$. (401)

$s_4 = R\sqrt{2}$. (396)

$s_6 = R$. (398)

$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. (339)

$m_c = \frac{1}{2}\sqrt{2(a^2+b^2)-c^2}$. (340, Ex. 5)

$t_c = \frac{2}{a+b}\sqrt{abs(s-c)}$. (341, ex. 6)

In similar polygons:

$p : p' = a : a'$ (318)

and $a : a' = b : b' = c : c'$, etc. (303)

$s_{2n} = \sqrt{2\,R^2 - R\sqrt{4\,R^2 - s_n^2}}$. (432)

An arc $= \frac{\text{central } \angle}{360^\circ} \times 2\,\pi R$. (425)

In circles,

$c : c' = d : d' = r : r'$. (421, 422)

## AREAS OF PLANE FIGURES

| | | |
|---|---|---|
| Rectangle, | $bh$. | (350) |
| Square, | $b^2$. | (351) |
| Parallelogram, | $bh$. | (353) |
| Triangle, | $\frac{1}{2}bh$, or $\sqrt{s(s-a)(s-b)(s-c)}$. | (358) |
| Equilateral △, | $\frac{b^2}{4}\sqrt{3}$. | (p. 228, Ex. 12) |
| Trapezoid, | $\frac{1}{2}h(b_1+b_2)$. | (363) |
| Rhombus, | $\frac{1}{2}d_1 d_2$. | (p. 228, Ex. 13) |
| Regular polygon, | $\frac{1}{2}pr$. | (426) |
| Circle, | $\pi R^2$ or $\frac{1}{2}Rc$. | (427, 428) |
| Sector of circle, | $\frac{1}{2}$ arc $\times R$ or $\frac{\text{central} \angle}{360} \times \pi R^2$. | (430) |
| Segment of ⊙, | Sector ± triangle formed by chord and radii drawn to the extremities of the chord. | |
| Similar polygons, | $S : S' = a^2 : a'^2 = p^2 : p'^2$. | (380, 382) |
| Two circles, | $S : S' = R^2 : R'^2 = d^2 : d'^2$. | (429) |

## TABLES

### Linear Measure

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
5½ yards = 1 rod
320 rods = 1 mile (mi.)
1 mile = 1760 yd. = 5280 ft.

### Square Measure

144 square in. = 1 sq. ft.
9 sq. ft. = 1 sq. yd.
30¼ sq. yd. = 1 sq. rod
160 sq. rods = 1 acre
640 acres = 1 sq. mi.

## METRIC SYSTEM

### Linear Measure

10 millimeters (mm.) = 1 centimeter (cm.)
10 centimeters = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
10 meters = 1 dekameter (dm.)
10 dekameters = 1 hectometer (hm.)
10 hectometers = 1 kilometer (km.)
1 meter = 39.37 in.

### Square Measure

100 sq. mm. = 1 sq. cm.
100 sq. cm. = 1 sq. dm.
100 sq. dm. = 1 sq. meter

### Approximations

$\sqrt{2} = 1.414$
$\sqrt{3} = 1.732$
$\sqrt{5} = 2.236$

$\pi = 3.1416$
$\pi = 3.14$
$\pi = 3\frac{1}{7}$

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